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Unitary Semi-Ring on Some Nexuses

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Abstract

This paper introduces and investigates a novel algebraic framework for structures composed of finite addresses, which we term *nexuses*. A nexus N is defined as a set of finite sequences (addresses) satisfying a closure property. We define its order, $n(N)$, as the maximum integer value appearing among its constituent addresses. Two custom binary operations, $\oplus$ and $\odot_n$, parameterized by a natural number n, are introduced. The central construction is the generation of a semi-ring $\langle N\rangle_n$ from a given finite-order nexus N, where $n = n(N)$. This generated structure is shown to be itself a nexus, thereby preserving the core combinatorial properties of the original set while enriching it with a coherent algebraic architecture. Notably, $\langle N\rangle_n$ is unitary, with multiplicative identity given by the unit address (1).

Keywords: Address, Nexus, Polynomial, Semi-ring, (prime) ideal, Unitary, Homomorphism, Zero-divisor.

1|Introduction

In 1980, Haristchain [6] introduced the concept of a *plenix* to handle large amounts of structured spatial data. In 1984, Nooshin [12] defined a *nexus* as a mathematical object that captures the constitution of a plenix, using the notion of an *address set* (also, see [7]). In 2009, Bolourian [2] introduced *nexus algebras* as an abstract algebraic structure and investigated their properties.

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Since then, many authors have studied nexuses and subnexuses. For instance, Norouzi [13] worked on subnexuses based on N-structures. Kamrani et al. [4, 5] developed the structure of a moduloid on a nexus and verified the concepts of submoduloids, finitely generated submoduloids, and prime submoduloids (also, see [3]). Recently, Nazemi Niya et al. [9, 10, 11] defined (fuzzy) semi-rings on a (finite) nexus N, characterized all prime ideals, and studied homomorphisms, quotients, and localizations.

However, the semi-rings constructed in those works are not unitary unless the nexus is cyclic. In this case, if $N = \langle a \rangle$ be a cyclic nexus, $1 = a$. In a non-unitary semi-ring, several standard properties fail. For example, the polynomial ring $R[x]$ over a non-unitary semi-ring does not contain the monomial x (since $x = 1 \cdot x$ would require a multiplicative identity). Moreover, units cannot be defined (let R be a unitary semi-ring with 1_R. An element $0 \neq a \in R$ is unite if there exists $b \in R$ such that $ab = ba = 1_R$).

In this paper, we construct a *unitary* semi-ring on certain nexuses where the multiplicative identity is exactly the address (1). For this purpose, we define the function $[\ ]_n$, where $n \in \mathbb{N}$ and the semi-ring $\mathbb{N}_n$. But this semi-ring is not unitary. Then by the semi-ring of polynomials on $\mathbb{N}_n$, i.e. $\mathbb{N}_n[x]$, we define two custom binary operations, $\oplus$ and $\odot_n$, parameterized by a natural number n. The central construction is the generation of a semi-ring $\langle N \rangle_n$ from a given finite-order nexus N, where $n = n(N)$. This generated structure is shown to be itself a nexus. Notably, $\langle N \rangle_n$ is unitary, with multiplicative identity given by the unit address (1). The nexus N with $n(N) = n$ is closed, if $\langle N \rangle_n = N$. Every closed nexus with $rise(N) \neq 1$ is an infinite nexus. We begin by defining the necessary operations and then verify the algebraic structures such as homorphism and isomorphism theorems, quotient semi-ring, localization semi-ring, Noetherian and Artinian properties, zero-divisor elements, idempotent elements and prime elements.

2|Preliminary Concepts

Now we recall the basic definitions ([8]).

A groupoid $(G, \circ)$ is a nonempty set with a binary operation. A semi-group is an associative groupoid. A monoid $(G, \circ, 0)$ is a semi-group with an identity element 0 such that $0 \circ a = a \circ 0 = a$ for all $a \in G$.

A semi-ring $(R, +, \circ, 0)$ is a nonempty set with two operations such that $(R, +, 0)$ is a commutative monoid, $(R, \circ)$ is a semi-group, and $\circ$ distributes over $+$: $a \circ (b + c) = a \circ b + a \circ c$ and $(b + c) \circ a = b \circ a + c \circ a$ for all $a, b, c \in R$. If there exists an element $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$, then R is called *unitary.* An element $a \in R$ is a unit if it has a multiplicative inverse.

A subset $X \subseteq R$ is a sub-semi-ring if $0 \in X$ and X is closed under $+$ and $\circ$. A subset $I \subseteq R$ is an ideal if $0 \in I$, $a + b \in I$ for all $a, b \in I$, and $ra, ar \in I$ for all $r \in R$ and $a \in I$. An ideal I is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$.

An *address* is a sequence whose elements belong to $\mathbb{N}^* = \{0\} \cup \mathbb{N}$, with the property that if $a_k = 0$ then $a_i = 0$ for all $i \geq k$. The empty sequence is denoted by $()$. A non-empty address is written as $(a_1, a_2, \ldots, a_n)$ where $a_i \in \mathbb{N}$. A *nexus* N is a non-empty set of addresses satisfying: $(a_1, \ldots, a_{n-1}, a_n) \in N$ implies $(a_1, \ldots, a_{n-1}, t) \in N$ for all $0 \leq t \leq a_n$.

For an infinite nexus $\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N}$ implies $\forall n \in \mathbb{N}, \forall\ 0 \leqslant t \leqslant a_n, (a_1, \cdots, a_{n-1}, t) \in N$. Hereafter, N denotes a nexus.

Let $a \in N$. The level of a is said to be:

(I) n, if $a = (a_1, a_2, \cdots, a_n)$, for some $a_k \in \mathbb{N}$,

(II) ∞, if a is an infinite sequence of N,

(III) 0, if $a = ()$.

The level of a is denoted by $l(a)$. If $a = (a_1, a_2, \cdots, a_n)$, we define *stem* $a = a_1$.

Let N be a nexus, we define $rise(N) = max\{l(a) \mid a \in N\}$[1]. In this paper, if a nexus N is a semi-ring with $rise(N) \neq 1$, then N is an infinite nexus.

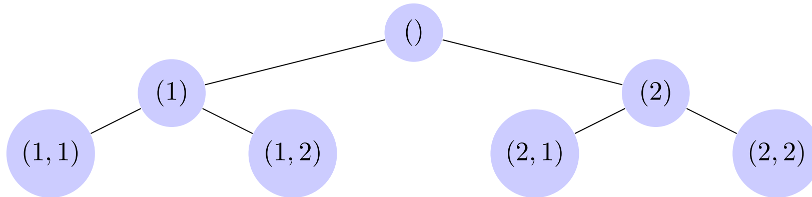

FIGURE 1. An example of a finite nexus $N = \{(), (1), (2), (1,1), (1,2), (2,1), (2,2)\}$.

3|Semi-ring on nexuses

For a finite nexus N, define $n(N) := \max\{a \in \mathbb{N} \mid \exists (a_0, \dots, a_k) \in N,\ \exists i\ (1 \le i \le k),\ \text{ such that } a_i = a\}$. We now introduce two binary operations on addresses.

Definition 1. For $a \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, define $[a]_n$ by $[a]_n = a$ if $0 \le a < n$, and $[a]_n = n$ if $a \ge n$.

Example 2. Let $n = 5$, then

$$[0]_5 = 0, [1]_5 = 1, [2]_5 = 2, [3]_5 = 3, [4]_5 = 4,$$
$$5 = [5]_5 = [6]_5 = [7]_5 = [8]_5 = \cdots .$$

Lemma 3. *If $a \le b$ then $[a]_n \le [b]_n$.*

Proof: Straightforward. □

Definition 4. For $a, b \in \mathbb{N}^*$, set $a + b = a \vee b = \max\{a, b\}$ and $a \cdot b = [ab]_n$. Denote this algebraic structure by $\mathbb{N}_n$.

Example 5. Let $2, 3 \in \mathbb{N}_4$, hence $2 + 3 = 3$ and $2 \cdot 3 = [2 \cdot 3]_4 = [6]_4 = 4$.

Let $2, 3 \in \mathbb{N}_7$, hence $2 + 3 = 3$ and $2 \cdot 3 = [2 \cdot 3]_7 = [6]_7 = 6$.

Theorem 6. *$(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.*

Proof: Clearly binary operations $+$ and $\cdot$ are well-defined and $+$ is associative and for every $a \in \mathbb{N}$, $a \vee 0 = 0 \vee a = a$. We show that $\cdot$ is associative. Let $a, b, c \in \mathbb{N}$. We consider two cases:

Case 1: Let $abc < n$. Hence $ab < n$ and $bc < n$. So $\big[[ab]_n c\big]_n = [abc]_n = abc$ and $\big[a[bc]_n\big]_n = [abc]_n = abc$. Then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Case 2: Let $abc \geqslant n$. If $a \geqslant n$, then $ab \geqslant n$ and so $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [nc]_n = n$. Also we have $a[bc]_n \geqslant n$ and so $a \cdot (b \cdot c) = \big[a[bc]_n\big]_n = n$. If $b \geqslant n$ or $c \geqslant n$, similarly we have $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. Let $a < n$, $b < n$ and $c < n$. If $ab \geqslant n$, we have $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [nc]_n = n$. Since $bc \geqslant b$, by Lemma 3, $[bc]_n \geqslant [b]_n = b$, hence $a[bc]_n \geqslant ab \geqslant n$, then $[a[bc]_n]_n = n$. Hence $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. If $bc \geqslant n$, similarly, $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$. Let $a < n$, $b < n$, $c < n$, $ab < n$ and $bc < n$. Hence $(a \cdot b) \cdot c = \big[[ab]_n c\big]_n = [abc]_n = n$ and $a \cdot (b \cdot c) = \big[a[bc]_n\big]_n = [abc]_n = n$. Therefore $(a \cdot b) \cdot c = a \cdot (b \cdot c) = n$ and . is associative.

Now we show that $a \cdot (b + c) = a \cdot b + a \cdot c$. Let $b \geqslant c$. So $ab \geqslant ac$ and by Lemma 3, $[ab]_n \geqslant [ac]_n$. Hence $a \cdot b + a \cdot c = [ab]_n \vee [ac]_n = [ab]_n = a \cdot b = a \cdot (b \vee c) = a \cdot (b + c)$. Similarly, $(b + c) \cdot a = b \cdot a + c \cdot a$. Therefore $\cdot$ on $+$ is distributive and so $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring. □

Hereafter, $\mathbb{N}_n$ is a semi-ring, it means, $(\mathbb{N}_n, +, \cdot, 0)$ is a semi-ring.

Corollary 7. *For $n > 1$, $\mathbb{N}_n$ is not unitary because for every $n < a \in \mathbb{N}$, we have $1 \cdot a = [a]_n = n \neq a$.*

Let $R[x]$ be the polynomial semi-ring over R. Define $R\lfloor x \rfloor = \{\sum_{i=0}^{m} a_i x^i \in R[x] \mid 0 \neq a_i \in R\} \cup \{0\}$.

Proposition 8. *If R is a semi-ring where $a + b \neq 0$ and $ab \neq 0$ for all nonzero $a, b \in R$, then $R\lfloor x \rfloor$ is a sub-semi-ring of $R[x]$.*

Proposition 9. *$\mathbb{N}_n\lfloor x\rfloor$ is a sub-semi-ring of $\mathbb{N}_n[x]$.*

Now we connect nexuses to polynomials.

Definition 10. Let M be the set of all finite addresses (including the empty address $()$). For $n \in \mathbb{N}$, define the map

$$\Phi_n : M \longrightarrow \mathbb{N}_n\lfloor x\rfloor$$

by

$$\Phi_n((a_0, a_1, \ldots, a_k)) \;=\; \sum_{i=0}^{k} a_i x^i,$$

with $\Phi_n(()) = 0$ (the zero polynomial). For addresses $a = (a_0, \ldots, a_k)$, $b = (b_0, \ldots, b_t)$ define:

$$a \oplus b = (a_0 \vee b_0,\ a_1 \vee b_1,\ \ldots,\ a_l \vee b_l), \quad l = k \vee t,$$

$$a \odot_n b = (c_0, c_1, \ldots, c_{k+t}), \quad c_i = \bigvee_{j=0}^{i} [a_{i-j} b_j]_n.$$

Example 11. Let $a = (1, 6, 3, 1)$, $b = (8, 4, 6)$ and $N = \langle a, b\rangle$, then $n(N) = 8$. Hence

$a \oplus b = \big((1 \vee 8), (6 \vee 4), (3 \vee 6), (1 \vee 0)\big) = (8, 6, 6, 1).$

$$\begin{aligned} a \odot_8 b &= \big((1.8), (1.4 + 6.8), (1.6 + 6.4 + 8.3), (1.0 + 6.6 + 3.4 + 1.8)\big) \\ &= \big([8]_8, ([4]_8 \vee [48]_8), ([6]_8 \vee [24]_8 \vee [24]_8), ([0]_8 \vee [36]_8 \vee [12]_8 \vee [8]_8)\big) \\ &= \big((8, (4 \vee 8), (6 \vee 8 \vee 8), (0 \vee 8 \vee 8 \vee 8)\big) \\ &= (8, 8, 8, 8). \end{aligned}$$

Proposition 12. *The map Φ_n is a bijection. Moreover, for all addresses $a, b \in M$,*

$$\Phi_n(a \oplus b) = \Phi_n(a) + \Phi_n(b), \qquad \Phi_n(a \odot_n b) = \Phi_n(a) \cdot \Phi_n(b).$$

Hence Φ_n is an isomorphism of semi-rings between $(M, \oplus, \odot_n, ())$ and $\mathbb{N}_n\lfloor x\rfloor$.

Proof: Injectivity is clear because different sequences yield different polynomials (with non-zero leading coefficients). Surjectivity follows because every polynomial $\sum_{i=0}^{k} a_i x^i$ comes from the address $(a_0, \ldots, a_k)$. The two addresses $()$ and (1) are exactly the definitions of $\oplus$ and $\odot_n$ (cf. Definition 10). Thus Φ_n preserves both operations and we have $() \leftrightarrow 0$ and $(1) \leftrightarrow 1$ (notice that M and $\mathbb{N}_n\lfloor x\rfloor$ aren't unitary). □

Theorem 13. *Let N be a nexus with $n(N) = n$ that is closed under $\oplus$ and $\odot_n$. Then $\big(N, \oplus, \odot_n, (), (1)\big)$ is a unitary semi-ring.*

Proof: Because $N \subseteq M$ and is closed under the operations and by Proposition 12, $(M, \oplus, \odot_n, ())$ is semi-ring, it inherits the semi-ring structure and so $\big(N, \oplus, \odot_n, (), (1)\big)$ is a semi-ring. Finally, for any $a = (a_0, \ldots, a_m) \in N$, we have $(1) \odot_n a = ([1 \cdot a_0]_n, \ldots, [1 \cdot a_m]_n) = (a_0, \ldots, a_m) = a$ because $a_i \leq n$. Hence (1) is the multiplicative identity. □

Definition 14. Let N be a nexus with $n(N) = n$. We define $\langle N\rangle_n$ the sub-nexus generated by N is closed under $\oplus$ and $\odot_n$.

Corollary 15. *Let N be a nexus with $n(N) = n$. Then $\big(\langle N\rangle_n, \oplus, \odot_n, (), (1)\big)$ is a unitary semi-ring.*

Proof: It is clear by Definition 14 and Theorem 13. □

Example 16. Let $N = \{(), (1)\}$. Then $n(N) = 1$ and $\langle N\rangle_1 = \{(), (1)\}$.

Example 17. Let $N = \{(), (1), (1, 1)\}$. Then $n(N) = 1$ and $\langle N\rangle_1$ contains all addresses of the form $(\underbrace{1, \ldots, 1}_{k})$ for $k \geq 0$.

Example 18. For $N = \{(), (1), (2), \ldots, (m)\}$, $n(N) = m$ and $\langle N\rangle_m = N$.

Example 19. Let $N = \{(), (1), (1,1), (1,2)\}$. Then $n(N) = 2$ and $\langle N \rangle_2$ consists of all addresses starting with 1 and whose entries are at most 2, with the nexus closure property.

Lemma 20. *Let N be a nexus with $n(N) = n$. For non-empty addresses $a, b \in \langle N \rangle_n$ with $l(a) = k \geq 1$ and $l(b) = m \geq 1$, we have $l(a \oplus b) = k \vee m$ and $l(a \odot_n b) = k + m - 1$. If either address is empty, the formulas are trivial.*

Proof: Let $a = (a_0, a_1, \cdots, a_{k-1})$ and $b = (b_0, b_1, \cdots, b_{m-1})$. By the proof of Theorem 13, $\Phi_n(a) = \sum_{i=0}^{k-1} a_i x^i$ is a polynomial of degree $k-1$ and $\Phi_n(b) = \sum_{i=0}^{m-1} b_i x^i$ is a polynomial of degree $m-1$. Hence $\Phi_n(a) + \Phi_n(b)$ is a polynomial of degree $(k \vee m) - 1$ and $\Phi_n(a)\Phi_n(b)$ is a polynomial of degree $k + m - 2$. Since $a \oplus b = \Phi_n^{-1}(\Phi_n(a) + \Phi_n(b))$ and $a \odot_n b = \varphi_n^{-1}(\Phi_n(a)\Phi_n(b))$, $l(a \oplus b) = k \vee m$ and $l(a \odot_n b) = k + m - 1$. □

Example 21. Let $a = (2,3,1)$ and $b = (3,5,2,1)$. By Lemma 20, $l(a \oplus b) = 4$ and $l(a \odot_5 b) = 3 + 4 - 1 = 6$. In fact, $a \oplus b = (3,5,2,1)$ and $a \odot_5 b = (2 + 3x + x^2)(3 + 5x + 2x^2 + x^3) = [6]_5 + [10 \vee 9]_5 x + [4 \vee 15 \vee 3]_5 x^2 + [2 \vee 6 \vee 5]_5 x^3 + [3 \vee 2 \vee 5]_5 x^4 + [1]_5 x^5 = (5,5,5,5,5,1)$.

Corollary 22. *Let N be a nexus with $n(N) = n$ and $rise(N) \neq 1$. Then $\langle N \rangle_n$ is infinite.*

Proof: It is clear by Lemma 20. □

Proposition 23. *Let $n(N) = n$ and $s \leq n$. Then $(1, s) \in \langle N \rangle_n$ iff $(1, s) \in N$.*

Proof: If $(1, s) \notin N$ but is in $\langle N \rangle_n$, by Lemma 20 it would be a product of an address of length 1 and an address of length 2, forcing $(1, s) \in N$ – contradiction. □

Proposition 24. *Let $s_1, s_2 \leq n$ and $(1, s_1) \in N$. Then $(1, s_1, s_2) \in \langle N \rangle_n$ iff either $(1, s_1, s_2) \in N$ or $s_2 \leq s_1^2$.*

Proof: If $(1, s_1, s_2) \in N$, clearly $(1, s_1, s_2) \in \langle N \rangle_n$. Now let $s_2 \leq s_1^2$. We have $(1, s_1) \odot_n (1, s_1) = (1, s_1, [s_1^2]_n) \in \langle N \rangle_n$. So for every $s_2 \leq s_1^2$, $(1, s_1, s_2) \in \langle N \rangle_n$. Conversely, let $(1, s_1, s_2) \in \langle N \rangle$. Since $l((1, s_1, s_2)) = 3$, by Lemma 20, there exist $a, b \in N$ with $l(a) = 2$ and $l(b) = 2$ such that $a \odot_n b = (1, s_1, s_2)$. Let $a = (a_0, a_1)$ and $b = (b_0, b_1)$. We have $a \odot_n b = ([a_0 b_0]_n, [a_0 b_1 \vee a_1 b_0]_n, [a_1 b_1]_n) = (1, s_1, s_2)$. Hence $a_0 = b_0 = 1$ and $a_1 \vee b_1 = s_1$ and $[a_1 b_1]_n = s_2$. So $a_1, b_1 \leq s_1$ and $s_2 = [a_1 b_1]_n \leq s_1^2$ and the proof is complete. □

Example 25. Let N be cyclic nexus $N = \big\langle (1, 10) \big\rangle$. By Proposition 24, we have:

(I) $(1,1,1) \in \langle N \rangle_{10}$, but for every $2 \leq j \leq 10$, $(1,1,j) \notin \langle N \rangle_{10}$.

(II) For every $1 \leq i \leq 4$, $(1,2,i) \in \langle N \rangle_{10}$, but for every $5 \leq i \leq 10$, $(1,2,i) \notin \langle N \rangle_{10}$.

(III) For every $1 \leq i \leq 9$, $(1,3,i) \in \langle N \rangle_{10}$, but $(1,3,10) \notin \langle N \rangle_{10}$.

(IV) For every $4 \leq k \leq 10$ and for every $1 \leq i \leq 10$, $(1,k,i) \in \langle N \rangle_{10}$.

Proposition 26. *$(1, s_1, s_2, s_3) \in \langle N \rangle_n$ is a product of two addresses iff there exist $s', s'' \in \mathbb{N}$ with $s', s'' \leq n$ such that one of the following conditions holds:*

(I) *$s' \leq s_1$ and $s'' \leq s'^2$ such that $s_2 = [s' s_1]_n$ and $s_3 = [s'' s_1]_n$.*

(II) *$s' \leq s_1$ and $s'' \leq s' s_1 \leq s_1^2$ such that $s_2 = [s' s_1]_n$ and $s_3 = [s' s'']_n$.*

(III) *$s' \leq s_1$ and $s' s_1 \leq s'' \leq s'^2$ such that $s_2 = s''$ and $s_3 = [s' s'']_n$.*

Proof: If part (I) holds. Since $s'' \leq s'^2$, by Proposition 24, $(1, s', s'') \in \langle N \rangle_n$. Now since $s'' \leq s'^2$, we have $s'' \leq s' s_1$ and $(1, s_1)(1, s', s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s_1 s'']_n) = (1, s_1, [s' s_1]_n, [s'' s_1]_n) \in \langle N \rangle_n$.

If part (II) holds. Since $s'' \leq s' s_1 \leq s_1^2$, by Proposition 24, $(1, s_1, s'') \in \langle N \rangle_n$. Now we have $(1, s')(1, s_1, s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s' s'']_n) = (1, s_1, [s' s_1]_n, [s' s'']_n) \in \langle N \rangle_n$.

If part (III) holds. Since $s' s_1 \leq s'' \leq s'^2$, by Proposition 24, $(1, s_1, s'') \in \langle N \rangle_n$. Now we have $(1, s')(1, s_1, s'') = (1, s_1 \vee s', [s' s_1 \vee s'']_n, [s' s'']_n) = (1, s_1, s'', s' s'') \in \langle N \rangle_n$.

Conversely, let $(1, s_1, s_2, s_3) \in \langle N \rangle_n$. Since $l((1, s_1, s_2, s_3)) = 4$, by Lemma 20, there exist $a, b \in N$ with $l(a) = 2$ and $l(b) = 3$ such that $a \odot b = (1, s_1, s_2, s_3)$. Let $a = (a_0, a_1)$ and $b = (b_0, b_1, b_2)$. We have

$a \odot b = (a_0b_0, [a_0b_1 \vee a_1b_0]_n, [a_0b_2 \vee a_1b_1]_n, [a_1b_2]_n) = (1, s_1, s_2, s_3)$. Hence $a_0 = b_0 = 1$ and $a_1 \vee b_1 = s_1$ and $[a_0b_2 \vee a_1b_1]_n = s_2$ and $[a_1b_2]_n = s_3$.

We consider three cases:

Case 1: $a_1 = s_1$ and $b_1 = s' \leq s_1$ and $b_2 = s'' \leq s'^2$. Since $s'' \leq s'^2$, $s'' \leq s's_1$. Hence $s_2 = [s'' \vee s's_1]_n = [s's_1]_n$ and $s_3 = [s''s_1]_n$.

Case 2: $a_1 = s' \leq s_1$ and $b_1 = s_1$ and $b_2 = s'' \leq s's_1 \leq s_1^2$. Hence $s_2 = [s'' \vee s's_1]_n = [s's_1]_n$ and $s_3 = [s's'']_n$.

Case 3: $a_1 = s' \leq s_1$ and $b_1 = s_1$ and $b_2 = s's_1 \leq s'' \leq s_1^2$. Hence $s_2 = [s'' \vee s's_1]_n = [s'']_n = s''$ and $s_3 = [s's'']_n$. □

Proposition 27. *If $(1, s_1, \ldots, s_k) \in \langle N \rangle_n$ then for each i, $s_i \leq s_1^i$.*

Proof: By induction using the polynomial representation: the degree-i coefficient in a product of polynomials of degrees r and t is bounded by $s_1^r \cdot s_1^t = s_1^{r+t-2}$ after appropriate normalization. □

Example 28. Let N be a nexus, $(1, 3) \in N$ and $n = 27$. All addresses of the form $(1, 3, s_2, s_3) \in \langle N \rangle_{27}$ are:

$$(1, 3, 3, i),\ 1 \leq i \leq 3,$$
$$(1, 3, 6, i),\ 1 \leq i \leq 12,$$
$$(1, 3, 9, i),\ 1 \leq i \leq 27,$$
$$(1, 3, 4, i),\ 1 \leq i \leq 4,$$
$$(1, 3, 5, i),\ 1 \leq i \leq 5,$$
$$(1, 3, 7, i),\ 1 \leq i \leq 14,$$
$$(1, 3, 8, i),\ 1 \leq i \leq 16.$$

The following example shows that the converse of Proposition 27 is not true.

Example 29. Let $N = \big\langle (1, 2) \big\rangle_2$, then $(1, 1, 2) \notin N$.

4|Ideals of semi-ring $\langle N \rangle_n$

Now we verify the ideals of semi-ring $\langle N \rangle_n$.

Definition 30. Let N be a nexus with $n(N) = n$. We say that N is a closed nexus if $\langle N \rangle_n = N$.

Hereafter, N denotes a closed nexus with $n(N) = n$.

Definition 31. Let N be a nexus. An ideal I of N satisfies $0 \in I$, $a \oplus b \in I$ for $a, b \in I$, and $ra \in I$ for $r \in N$, $a \in I$.

Given $a \in N$, the principal ideal generated by a is $\langle a \rangle_n = \{b \odot_n a \mid b \in N\}$.

Example 32. Let $s \in \mathbb{N}$, $N = \big\langle (1, s) \big\rangle$, then $n(N) = s$. If I is the ideal generated by $(1, s)$ of semi-ring $\langle N \rangle_s$, then $I = \big\{(1, s), (1, s, s), (1, s, s, s), \cdots \big\} \cup \{()\}$.

Definition 33. Let $a = (a_1, \cdots, a_k), b = (b_1, \cdots, b_t)$ be two addresses with $k \leqslant t$. We say $a \leqslant b$, if for every $i(1 \leqslant i \leqslant t)$, $a_i \leqslant b_i$. Clearly $a \leqslant b$ if and only if $a \oplus b = b$.

Example 34. Let $a = (3, 3)$ and $b = (3, 3, 3)$, then $a \leqslant b$.

Let $a = (3, 2)$ and $b = (3, 3)$, then $a \leqslant b$.

Let $a = (3, 1)$ and $b = (3, 3, 3)$, then $a \leqslant b$.

Let $a = (3, 3)$ and $b = (2, 3, 3)$, then $a \nleqslant b$ and $b \nleqslant a$

Definition 35. Let N be a nexus and $a \in N$. We put $B_a = \{b \in N \mid b \geqslant a\} \cup \{0\}$. It is easy to see that B_a is an ideal of N.

Proposition 36. *Suppose that $N = \langle(1,1)\rangle_1$ and I be the ideal generated by $(\underbrace{1,\cdots,1}_{n\ times})$ with $n \geq 2$. Then $I = \{()\} \bigcup \{(\underbrace{1,\cdots,1}_{k\ times}) : k \geqslant n\}$ is prime if and only if $n = 2$.*

Remark 37. Let $M = \langle a \rangle$ be a cyclic nexus with $n(M) = n$. Then the nexus $N = \langle M \rangle_n$ is a closed nexus. We denoted this nexus by N_a. Let $s \in \mathbb{N}$ and $a = (1, s)$. Then $n(M) = s$ and we denoted the nexus N_a by N_s and B_a by B_s.

Proposition 38. *For every $s \in \mathbb{N}$, B_s is a prime ideal.*

Proof: It is easy to see that if $a \in B_s - \{(), (1, s)\}$, there exist $s_1, \cdots, s_k \in \mathbb{N}$, such that $a = (1, s, s_1, s_2, \cdots, s_k)$. Now let $b, c \in N$ and $(a_1, a_2, \cdots, a_m) = b \odot_s c \in B_s$. Then there exist $r, t \in \mathbb{N}$ with $r + t - 1 = m$ and $b_1, \cdots, b_r, c_1, \cdots, c_t \in \mathbb{N}$, such that $b = (b_1, \cdots, b_r)$ and $c = (c_1, \cdots, c_t)$. Clearly $b_1 = c_1 = 1$. If $b_2, c_2 < s$, then $a_2 < s$ - contradiction. Hence $b_2 = s$ or $c_2 = s$ and so $b \in B_s$ or $c \in B_s$. Therefore B_s is a prime ideal. $\square$

The following example shows that for a nexus N and $a \in N$, B_a is not prime ideal.

Example 39. For the nexus $N_{(3,3)}$, we have $(3, 2) \odot_3 (3, 2) = (3, 3, 3) \in B_{(3,3)}$, but $(3, 2) \notin B_{(3,3)}$.

Proposition 40. *For every $2 \leqslant t \leqslant s \in \mathbb{N}$, the ideal $B_{(s,s)}$ of nexus $N_{(s,t)}$ is not prime.*

Proof: Case 1: Let $t \neq s$. There exist $k \in \mathbb{N}$ such that $(s,t)^k = (\underbrace{s,\cdots,s}_{(k+1)\text{ times}}) \in B_{(s,s)}$, but $(s,t) \notin B_{(s,s)}$.

Case 2: Let $s = t$, so $(s) \odot_s (s, 1) = (s, s) \in B_{(s,s)}$, but $(s), (s, 1) \notin B_{(s,s)}$. $\square$

In the followig proposition, we generalize Proposition 40.

Proposition 41. *Let $a = (s_1, \cdots, s_t)$ with $s_1 \neq 1$ and $s = max\{s_1, \cdots, s_t\}$. Then the ideal $B_{(\underbrace{s,\cdots,s}_{t\ times})}$ of nexus N_a is not prime.*

Proof: Since $s_1 \neq 1$, (s) and $(\underbrace{s,\cdots,s}_{t\text{ times}}) \in N_a$. If there exist $i(1 \leqslant i \leqslant t)$, such that $s_i \neq s$, hence $(s) \odot_s (s_1, \cdots, s_t) = (\underbrace{s,\cdots,s}_{t\text{ times}}) \in B_{(\underbrace{s,\cdots,s}_{t\text{ times}})}$, but $(s), (s_1, \cdots, s_t) \notin B_{(\underbrace{s,\cdots,s}_{t\text{ times}})}$. If for every $i(1 \leqslant i \leqslant t)$, such that $s_i = s$, hence $(s) \odot_s (\underbrace{s,\cdots,s}_{t-1\text{ times}}, 1) = (\underbrace{s,\cdots,s}_{t\text{ times}}) \in B_{(\underbrace{s,\cdots,s}_{t\text{ times}})}$, but $(s), (\underbrace{s,\cdots,s}_{t-1\text{ times}}, 1) \notin B_{(\underbrace{s,\cdots,s}_{t\text{ times}})}$. Therefore the ideal $B_{(\underbrace{s,\cdots,s}_{t\text{ times}})}$ is not prime. $\square$

In the followig proposition, we show that we can't generalize Propposition 40.

Proposition 42. *Let $a = (1, \underbrace{s,\cdots,s}_{t\ times})$. Then the ideal $B_{(1,\underbrace{s,\cdots,s}_{t\ times})}$ of nexus N_a is prime iff $t = 1$.*

Proof: Let $t = 1$, by Proposition 38, B_s is prime ideal. Conversely let $t \neq 1$, then $(1, s) \odot_s (1, \underbrace{s,\cdots,s}_{(t-1)\text{ times}}) = (1, \underbrace{s,\cdots,s}_{t\text{ times}}) \in B_{(1,\underbrace{s,\cdots,s}_{t\text{ times}})}$, but $(s), (1, \underbrace{s,\cdots,s}_{(t-1)\text{ times}}) \notin B_{(1,\underbrace{s,\cdots,s}_{t\text{ times}})}$. Therefore $B_{(1,\underbrace{s,\cdots,s}_{t\text{ times}})}$ is not prime ideal. $\square$

Proposition 43. *Let N be a nexus and $a, b \in N$. Then*

(I) $B_{(a\oplus b)} = B_a \cap B_b$.

(II) $B_{a\odot_n b} \subseteq B_a \cap B_b$.

Proof: (I) Let $x \in B_{(a\oplus b)}$. Since $x \geqslant a\oplus b$, $x \geqslant a, b$. So $x \in B_a$ and $x \in B_b$. Hence $B_{(a\oplus b)} \subseteq B_a \cap B_b$. Conversely let $x \in (B_a \cap B_b)$. Since $x \geqslant a, b$, $x \geqslant a \oplus b$. So $x \in B_a \cap B_b$. Therefore $B_{(a\oplus b)} = B_a \cap B_b$.

(II) Let $x \in B_{(a\odot_n b)}$. Since $x \geqslant a \odot_n b$, $x \geqslant a, b$. So $x \in B_a$ and $x \in B_b$. Hence $B_{(a\odot_n b)} \subseteq B_a \cap B_b$. ☐

The following example shows that the converse of Proposition 43 is not true.

Example 44. Let $N = N_{(6,3)}$, $a = (6, 2)$ and $b = (6, 3)$. We have $a \odot_6 b = (6, 6, 6)$. Hence $(6, 6, 6) \in B_{(6,6,6)}$ but $(6, 6, 6) \notin B_{(6,2)}$ and $(6, 6, 6) \notin B_{(6,3)}$.

Definition 45. For an ideal I and $a \in N$, define $a + I = \{a \oplus b \mid b \in I\}$.

Lemma 46. *$a + I = I$ iff $I \subseteq B_a$.*

Proof: Let $I \subseteq B_a$. Suppose that $x \in a + I$, then there exists $c \in I$ suth that $x = a \oplus c$. Since $c \geqslant a$, $x = a \oplus c = c \in I$. Hence $a + I \subseteq I$. Now let $x \in I$. Since $x \geqslant a$, $x = a \oplus x \in a + I$. Hence $I \subseteq a + I$ and so $a + I = I$. Conversely let $a + I = I$ and $x \in I$. Then there exist $c \in I$ such that $x = a \oplus c \geqslant a$ and so $x \in B_a$. Therefore $I \subseteq B_a$. ☐

Corollary 47. *Let N be a nexus and $a \in N$. Then*

(I) $a + B_a = B_a$;

(II) $a + I \subseteq B_a$.

Proposition 48. *Let N be a nexus, I be an ideal and $a, b \in N$. Hence $a + I = b + I$ iff $a = b$.*

Proof: By Corollary 47, $a + I = b + I$ iff $B_a = B_b$ iff $a = b$. ☐

Proposition 49. *Let N be a nexus and $a, b \in N$. Then $a + B_b = b + B_a = B_{(a\oplus b)}$.*

Proof: Let $x \in a + B_b$, so there exist $c \in B_b$ such that $x = a \oplus c$. Since $c \geqslant b$, $a \oplus c \geqslant a \oplus b$ and so $x \in B_{(a\oplus b)}$. Hence $a + B_b \subseteq B_{(a\oplus b)}$. Conversely let $x \in B_{(a\oplus b)}$. Since $x \geqslant a \oplus b$, $x \geqslant a, b$. So $x \in B_b$ and $x = a \oplus x \in a + B_b$. Hence $B_{(a\oplus b)} = a + B_b$. Similarly, $b + B_a = B_{(a\oplus b)}$. Therefore $a + B_b = b + B_a = B_{(a\oplus b)}$. ☐

Definition 50. Define $\frac{N}{I} = \{a + I \mid a \in N\}$ with $\boxplus$ defined by $(a + I) \boxplus (b + I) = (a \oplus b) + I$ and $\boxdot$ defined by $(a + I) \boxdot (b + I) = (a \odot_n b) + I$.

Proposition 51. *$(\frac{N}{I}, \boxplus, \boxdot, I, (1) + I)$ is a semi-ring.*

Proof: It is clear by Proposition 48. ☐

Corollary 52. *Let N be a nexus, I be an ideal of N. Then $\dfrac{N}{I} \cong N$.*

Proof: It is clear by Proposition 48. ☐

Further Algebraic Properties

In this section we deepen the algebraic study of the unitary semi-ring $\langle N\rangle_n$ by investigating homomorphisms, localization structures, and several additional ring-theoretic properties. These results enrich the theory and reveal strong connections with classical polynomial semi-rings.

Homomorphisms and Isomorphism Theorems. We begin by establishing a fundamental isomorphism between the set of all finite addresses and the polynomial semi-ring over $\mathbb{N}_n$.

Corollary 53. *Every semi-ring $\langle N\rangle_n$ (where $N \subseteq M$ is closed under the operations) is isomorphic to a sub-semi-ring of $\mathbb{N}_n[x]$. In particular, $\langle N\rangle_n$ is commutative (since $\mathbb{N}_n[x]$ is commutative).*

Proof: The restriction of Φ_n to $\langle N\rangle_n$ gives an injective semi-ring homomorphism onto its image. □

Now we turn to homomorphisms between different nexus semi-rings.

Definition 54. Let N, M be nexuses with $n(N) = n$ and $n(M) = m$, and let $\langle N\rangle_n$, $\langle M\rangle_m$ be their generated unitary semi-rings. A map $f : \langle N\rangle_n \to \langle M\rangle_m$ is a *semi-ring homomorphism* if it preserves $\oplus$, $\odot_n$, and the identity (1) (and sends () to ()).

Proposition 55 (First Isomorphism Theorem)**.** *Let $f : \langle N\rangle_n \to \langle M\rangle_m$ be a surjective semi-ring homomorphism. Define the kernel*

$$\ker f = \{a \in \langle N\rangle_n \mid f(a) = ()\}.$$

Then $\ker f$ is an ideal of $\langle N\rangle_n$, and the quotient $\langle N\rangle_n/\ker f$ (whenever the quotient is well-defined) is isomorphic to $\langle M\rangle_m$.

Proof: The proof is standard: one checks that $\ker f$ is an ideal, and the induced map $\bar{f} : \langle N\rangle_n/\ker f \to \langle M\rangle_m$ given by $\bar{f}(a + \ker f) = f(a)$ is well-defined, bijective, and preserves operations. The well-definedness of the quotient multiplication is a delicate point, but for the purpose of this theorem we assume it holds (see the next subsection for conditions). □

Noetherian and Artinian properties.

Definition 56. A semi-ring R is *Noetherian* if every ascending chain of ideals stabilises and is *Artinian* if every decending chain of ideals stabilises.

Hereafter, N denotes a closed nexus with $n(N) = n$.

We now turn to the property of being Noetherian and Artinian.

Proposition 57. *Let N be a nexus N, then N is Noetherian.*

Proof: Since N is isomorphic to a sub-semi-ring of $\mathbb{N}_n[x]$, and $\mathbb{N}_n$ is Noetherian semi-ring, the polynomial semi-ring $\mathbb{N}_n[x]$ is Noetherian by the Hilbert basis theorem for semi-rings (which holds for finite coefficient semi-rings). Subsemirings of Noetherian semi-rings are Noetherian. Hence N is Noetherian. □

Remark 58. For a nexus N, the Artinian property is not hold in general. See the following example.

Example 59. Let $N = \langle(1,1)\rangle_1$ we have the following chain of ideals:

$$\langle(1,1)\rangle \supsetneqq \langle(1,1,1)\rangle \supsetneqq \langle(1,1,1,1)\rangle \supsetneqq \cdots$$

.

Zero-divisors and Idempotents. We examine zero-divisors and idempotents.

Definition 60. A non-zero element $a \in N$ is a *zero-divisor* if there exists non-zero b such that $a \odot_n b = ()$ or $b \odot_n a = ()$.

Proposition 61. *The semi-ring N has no zero-divisors.*

Proof: Let $a, b \neq ()$. By Lemma 19, $l(a \odot_n b) = l(a) + l(b) - 1 \geq 1$, so $a \odot_n b \neq ()$. Thus no non-zero elements multiply to zero. □

Remark 62. This is a strong property: all N are integral domains (in the semiring sense). This is unusual for semi-rings built from finite sets, but it follows from the length additivity of multiplication.

Definition 63. An element $a \in N$ is *idempotent* if $a \odot_n a = a$.

Proposition 64. *The idempotent elements of N are precisely (1) and, if $(n) \in N$, also (n).*

Proof: If a is idempotent and $a \neq ()$, then $l(a) = 1$ by length considerations. Thus $a = (t)$ with $1 \leq t \leq n$. The condition $(t) \odot_n (t) = (t)$ gives $[t^2]_n = t$. Solving yields $t = 1$ or $t = n$ (if $t^2 \geq n$). Thus the claimed set. □

prime elements.

Definition 65. Let N be a nexus and $a, b \in N$. We say that $a \mid b$, if there exists $c \in N$ such that $a \odot_n c = b$.

Definition 66. Let N be a nexus and $a \in N$. We say that a is prime element of N, if $a = b \odot_n c$ implies that $b = (1)$ or $c = (1)$.

Proposition 67. *Let N be a nexus. Then the address $(k) \in N$, is prime iff $k \in \mathbb{N}$ is prime and $2 \leqslant k \lneqq n$;*

Proof: By the definition of prime element of $\mathbb{N}$ it is clear. Notice that (n) is not prime, becuase $(n) \odot_n (n) = (n)$ and $(n) \neq (1)$. □

Proposition 68. *Let N be a nexus of all addresses $a = (a_1, \cdots, a_k)$ such that for every $1 \leqslant i \leqslant k$, $1 \leqslant a_i \leqslant n$. If $GCD(a_1, \cdots, a_k) \neq 1$, then a is not prime.*

Proof: Let $1 \neq d = GCD(a_1, \cdots, a_k)$. Hence there exist $c_1, \cdots, c_k \in \mathbb{N}$ such that $a_i = c_i d (1 \leqslant i \leqslant k)$. So $(d) \odot_n (c_1, \cdots, c_k) = (a_1, \cdots, a_k)$ and hence a is not prime. □

The following example shows that the converse of Proposition 68(II), is not true.

Example 69. (I) Let $n = 7$ and $a = (2, 7)$. Then $(2, 7) = (2) \odot_7 (1, 4)$. Hence $(2, 7)$ is not prime, but $GCD(2, 7) = 1$.

(II) Let $a = (1, 1, 1)$, then $(1, 1, 1) = (1, 1) \odot_n (1, 1)$. Hence $(1, 1, 1)$ is not prime, but $GCD(1, 1, 1) = 1$.

localization of semi-ring. Let N be nexus and $S \subseteq N$. We say that S is a multiplicatively closed subset (m.c.s) of N, if $() \notin S$, $(1) \in S$ and for every $a, b \in S$, $a \odot_n b \in S$.

Now, we define a relation on $N \times S$. For every $a, a' \in N$ and $s, s' \in S$, $(a, s) \sim (a', s')$, iff there exist $t \in S$ such that $t \odot_n s' \odot_n a = t \odot_n s \odot_n a'$. It is easy to see that $\sim$ is an equivalence relation on $N \times S$.

For every $(a, s) \in N \times S$, put $\dfrac{a}{s} = \{(a', s') : (a, s) \sim (a', s')\}$. Also we define $N_S = \{\dfrac{a}{s} : a \in N \text{ and } s \in S\}$.

Definition 70. Let N be a nexus and $S \subseteq N$ be an (m.c.s) of N. We define the binary operations $\circledast$ and $\circledcirc$ on N_S with the following:

$$\frac{a}{s} \circledast \frac{b}{t} = \frac{(t \odot_n a) \oplus (s \odot_n b)}{s \odot_n t} \text{ and } \frac{a}{s} \circledcirc \frac{b}{t} = \frac{a \odot_n b}{s \odot_n t}.$$

Proposition 71. *The algebra $(N_S, \circledast, \circledcirc, \dfrac{()}{(1)}, \dfrac{(1)}{(1)})$ is a unitary semi-ring.*

Proof: Let $a, a', b, b' \in N$, $s, s', t, t' \in S$ and $\dfrac{a}{s} = \dfrac{a'}{s'}$ and $\dfrac{b}{t} = \dfrac{b'}{t'}$. Then there exist $s'', t'' \in S$ such that $s'' \odot_n s' \odot_n a = s'' \odot_n s \odot_n a'$ and $t'' \odot_n t' \odot_n b = t'' \odot_n t \odot_n b'$. Let $u = s'' \odot_n t''$, it is easy to see that $u \odot_n (s' \odot_n t')(t \odot_n a \oplus s \odot_n b) = u \odot_n (s \odot_n t)(t' \odot_n a' \oplus s' \odot_n b')$ and $u \odot_n t' \odot_n s' \odot_n a \odot_n b = u \odot_n t \odot_n s \odot_n a' \odot_n b'$. Hence N_S is closed under the operations $\circledast$ and $\circledcirc$ and it inherits the semi-ring structure of the semi-ring N, Also for every $a \in N$ we have

$$\frac{a}{s} \circledast \frac{()}{(1)} = \frac{a}{s} \text{ and } \frac{a}{s} \circledcirc \frac{(1)}{(1)} = \frac{a}{s}.$$

Thereore the algebra $(N_s, \circledast, \circledcirc, \dfrac{()}{(1)}, \dfrac{(1)}{(1)})$ is a unitary semi-ring. □

Example 72. Let $N = \langle(1,1)\rangle_1$ and $S = N - \{()\}$. Clearly S is an (m.c.s) of N. We have

$\frac{1}{\underbrace{(1,\cdots,1)}_{k \text{ times}}} = \frac{1}{\underbrace{(1,\cdots,1)}_{m \text{ times}}}$ iff $k = m$. If $m \geqslant k$ then, $\frac{\overbrace{(1,\cdots,1)}^{k \text{ times}}}{\underbrace{(1,\cdots,1)}_{m \text{ times}}} = \frac{1}{(\underbrace{1,\cdots,1}_{(m-k+1) \text{ times}})}$. Hence

$$N_S = \{\frac{()}{(1)}\} \cup \{\frac{\overbrace{(1,\cdots,1)}^{k \text{ times}}}{1} : k \geqslant 1\} \cup \{\frac{1}{\underbrace{(1,\cdots,1)}_{k \text{ times}}} : k \geqslant 2\}$$

.

This completes our additional algebraic investigation. The results obtained here complement the earlier structural theorems and pave the way for future work on ideals, homomorphisms, and applications.

4|Conclusion

We have shown that the semi-ring $\mathbb{N}_n$ is not unitary, but by embedding a suitable nexus into the polynomial semi-ring $\mathbb{N}_n\lfloor x\rfloor$, we obtain a unitary semi-ring $\langle N\rangle_n$ with identity (1). This construction opens the door to further study of algebraic properties (ideals, homomorphisms, localizations) in a framework that combines combinatorics of addresses with ring theory.

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Author Contribution

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References

[1] Taba, D. A., Hasankhani, A., & Bolurian, M. (2012). Soft nexuses. *Computers & mathematics with applications, 64*(6), 1812-1821. https://doi.org/10.1016/j.camwa.2012.02.048

[2] Bolourian, M. (2009). *Theory of plenices* [Thesis]

[3] Bolourian, M., Kamrani, R., & Hasankhani, A. (2019). The structure of moduloid on a nexus. *Mathematics, 7*(1), 1-16. https://doi.org/10.3390/math7010082

[4] Kamrani, R., Hasankhani, A., & Bolourian, M. (2020). Finitely generated submoduloids and prime submoduloids on a nexus. *Missouri journal of mathematical sciences, 32*(1), 88-109. https://doi.org/10.35834/2020/3201088

[5] Kamranialiabad, R., Hasankhani, A., & Bolourian, M. (2020). On submoduloids of a moduloid on nexus. *Applications and applied mathematics: An international journal (AAM), 15*(2), 42. https://digitalcommons.pvamu.edu/aam/vol15/iss2/42/

[6] Haristchian, M. (1980). *Formex and plenix structural analysis* [Thesis].

[7] Hee, I. (1985). *Plenix structural analysis* [Thesis].

[8] Hungerford, T. W.(1989). *Algebra, graduate texts in mathematics, Vol. 73*. Springer-Verlag, New York. https://books.google.com/books/about/Algebra.html?id=t6N_tOQhafoC

[9] Niya, V. N., Babaei, H., & Rezaei, A. (2024). On fuzzy sub-semi-rings of nexuses. *AIMS Mathematics, 9*(12), 36140-36157. https://doi.org/10.3934/math.20241715

[10] Nazemi Niya, V., Babaei, H., & Rezaei, A. (2026). On prime ideals on a semi-ring associated with a nexus. *Algebraic structures and their applications, 13*(3), 229-248. https://doi.org/10.22034/as.2025.23198.1796

[11] Nazemi Niya, V. & Rezaei, A. (2025). *Unitary semi-ring on finite nexuses* [Presentation]. Proceedings of the 12th national mathematics conference of payame noor university (pp. 340–343).

[12] Nooshin, H. (1984). *Formex configuration processing in structural engineering*. Elsevier Applied Science Publishers: London, UK. https://books.google.com/books/about/Formex_Configuration_Processing_in_Struc.html?id=e9LtAAAAIAAJ

[13] Norouzi, M., Asadi, A., & Jun, Y. B. (2018). Subnexuses based on n-structures. *Armenian journal of mathematics*, *10*(10), 1-15. https://doi.org/10.52737/18291163-2018.10.10-1-15