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Certain Notions of Energy in Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

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Abstract

Fermatean Quadripartitioned Neutrosophic fuzzy graph (FQNFG) is the integrating form of Fermatean and Quadripartitioned Neutrosophic fuzzy graph. Graph energy is recognized as a crucial concept in fuzzy graph theory for its ability to handle random events, thus capturing the attention of numerous researchers. Moreover, the study of graph energy has been a notable rise in recent years. Energy of Graphs have significant applications in various domains, including network analysis, decision making, Image processing, modelling uncertainty etc. This paper introduces energy and Laplacian energy for FQNFG. Adjacency matrix, eigen values, energy and Laplacian energy of FQNFG are defined with examples. Furthermore, we obtain lower and upper bounds of energy and Laplacian energy for FQNFG. Additionally, this study represents a sophisticated decision-making framework, employing a scoring methodology to assess and compare Laptops based on critical attributes such as processing power, memory & storage, Battery life and Display quality.

Keywords: Neutrosophic fuzzy graph, Quadripartitioned Neutrosophic fuzzy graph, Fermatean Quadripartitioned Neutrosophic fuzzy graph, Eigen Values, Energy of FQNFG, Laplacian energy of FQNFG, Multi-criteria decision making.

1 | Introduction

Fuzzy Set Theory concept was initiated by Zadeh (1965) [19]. Fuzzy graph theory merges the concepts of fuzzy sets with the structure of graphs and has widespread applications in various fields. Fuzzy graphs helps in modelling and analyzing the complex systems with uncertain or imprecise relationships. In 1975, Rosenfeld created the theory of Fuzzy Graphs. [14]. An idea about intutionistic fuzzy set relationships was initially introduced by Attanssov. They have putforth a number of applications, theorems and properties [16].

The classical, fuzzy and intuitionistic fuzzy sets were basic extensions of neutrosophic set. Smarandache generalized the fuzzy set, leads to development of the neutrosophic set [18]. It can handle any real world issue with ambiguous, inconsistent, uncertain and indeterminate data. Every item in a neutrosophic set has three membership grades: truth, indeterminacy and false. These three membership tiers fall between [0, 1] and are always independent.

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Senapati et al. (2020) [15] presented the novel idea called Fermatean Fuzzy Set. Based on the Fermatean Neutrosophic fuzzy set, Said Broumi et al. (2022) [2] presented a novel framework for Fermatean neutrosophic graphs and their applications. Later V. Divya and J. Jesintha Rosline presented an innovative concept known as the Fermatean Quadripartitioned neutrosophic Fuzzy Graph. This Graph helps in evaluating complex decisions with multiple criteria and uncertain outcomes.
I.Gutman [4] define "Graph Energy" as sum of the magnitudes of eigen values of the adjacency matrix in graph. The energy bounds are covered in [3, 5, 8]. Energy of different graphs like regular, non-regular and circulant are studied in [6, 7, 17]. The graph's energy is expanded to energy of fuzzy graph [9]. Later, energy of fuzzy graph broadened to intuitionistic and Neutrosophic graph [12, 10, 11]. The energy is applied to picture fuzzy graph, Pythagorean Fuzzy Graph, single valued neutrosophic fuzzy graph etc. Fuzzy graph Laplacian energy was introduced by Sadegh Rahimi Sharbaf et al. (2014) [13]. Later this was extended to Intuitionistic and Neutrosophic fuzzy graph [1, 10]. The energy of fuzzy graph has found applications in various fields like Chemical Graph Theory, Social Network analysis, Decision Making under uncertainty. The ability to model and analyse these uncertainties using fuzzy graph energy offers valuable insights and tools for decision making analysis and understand complex systems.
This study examines the energy and Laplacian energy of FQNFG. The paper is structured as follows. Section 2 presents prelimnaries . Section 3 defines energy of FQNFG. The lower and upper bounds of energy for FQNFG were also derived. In Section 4, we define and characterize Laplacian energy of FQNFG. Section 5 presents an application of determining the best Laptop involving four characteristics employing multi-criteria decision-making technique. Section 6 ends with a discussion of further research.

2 | Preliminaries

[2] A Fermatean Neutrosophic Graph (FNG) on a universal set X is a pair $G = (P, Q)$ where P is Fermatean neutrosophic set on X and Q is a Fermatean Neutrosophic relation on X so that:

$$\begin{aligned} T_Q(u,v) &\leq min(T_p(u), T_p(v)) \\ I_Q(u,v) &\geq max(I_p(u), I_p(v)) \\ F_Q(u,v) &\geq max(F_p(u), F_p(v)) \end{aligned}$$

with $0 \leq T_Q^3(u,v) + I_Q^3(u,v) + F_Q^3(u,v) \leq 2\ \forall u,v \in X$, such that, $T_Q : X \times X \rightarrow [0,1]$, $I_Q : X \times X \rightarrow [0,1]$ and $F_Q : X \times X \rightarrow [0,1]$ indicates degree of membership, indeterminacy-membership and non-membership of Q, Correspondingly, where P and Q are Fermatean Neutrosophic vertex and edge set of G. [9] Consider a fuzzy graph, $G = (V, \sigma, \mu)$ with adjacency matrix A. The energy G is termed as sum of the magnitudes of its eigenvalues. [1] Consider a fuzzy graph, $G = (\sigma, \mu)$ with $|V| = n$ vertices and $\mu_1 \geq and \mu_2 \geq \cdots \mu_n$ are Laplacian eigen values of $G = (\sigma, \mu)$. Laplacian energy of G is denoted by

$$LE(G) = \left| \mu_i - \frac{2 \sum\limits_{1 \leq i \leq j \leq n} \mu(u_i u_j)}{n} \right|$$

A Fermatean Quadripartitioned Neutrosophic Fuzzy Graph (FQNFG) an Universal Set X is a pair $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$, $\sigma_{FQ} : X \rightarrow [0,1]$ represents Fermatean Quadripartitioned Neutrosophic Set on X and $\mu_{FQ} : X \times X \rightarrow [0,1]$ denotes Fermatean Quadripartitioned Neutrosophic mapping on $X \times X$ so that

$$\begin{aligned} T_{\mu_{FQ}}(uv) &\leq \min(T_{\sigma_{FQ}}(u), T_{\sigma_{FQ}}(v)) \\ C_{\mu_{FQ}}(uv) &\leq \min(C_{\sigma_{FQ}}(u), C_{\sigma_{FQ}}(v)) \\ U_{\mu_{FQ}}(uv) &\geq \max(U_{\sigma_{FQ}}(u), U_{\sigma_{FQ}}(v)) \\ F_{\mu_{FQ}}(uv) &\geq \max(F_{\sigma_{FQ}}(u), F_{\sigma_{FQ}}(v)) \end{aligned}$$

with $0 \leq T^3_{\mu_{FQ}}(uv) + C^3_{\mu_{FQ}}(uv) + U^3_{\mu_{FQ}}(uv) + F^3_{\mu_{FQ}}(uv) \leq 3\ \forall u,v \in X$, where $T_{\mu_{FQ}} : XxX \rightarrow [0,1]$, $C_{\mu_{FQ}} : X \times X \rightarrow [0,1]$, $U_{\mu_{FQ}} : X \times X \rightarrow [0,1]$, $F_{\mu_{FQ}} : X \times X \rightarrow [0,1]$, indicates degree of truth, contradiction, ignorance

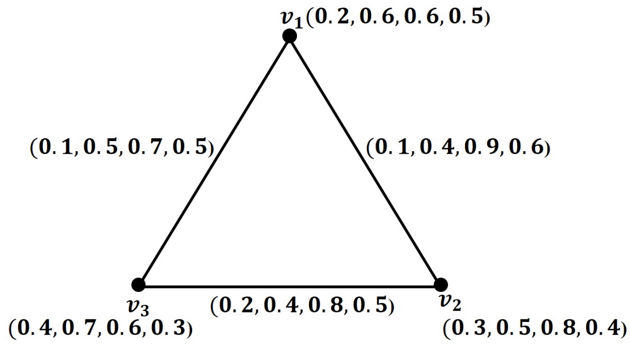

FIGURE 1. Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

and false membership of μ_{FQ}. σ_{FQ} and μ_{FQ} indicates Fermatean Quadripartitioned Neutrosophic vertex and edge set of G_{FQ}.

3 | Energy in Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

The Adjacency matrix $A(G_{FQ})$ of FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is termed as square matrix $A(G_{FQ}) = [a_{ij}]$, $a_{ij}=\left(T_{\mu_{FQ}}(p_ip_j), C_{\mu_{FQ}}(p_ip_j), I_{\mu_{FQ}}(p_ip_j), F_{\mu_{FQ}}(p_ip_j)\right)$ where $T_{\mu_{FQ}}(p_ip_j)$, $C_{\mu_{FQ}}(p_ip_j)$, $I_{\mu_{FQ}}(p_ip_j)$ and $F_{\mu_{FQ}}(p_ip_j)$ indicates the strength of relationship, undecided relationship, unknown relationship and non-relationship of p_i and p_j correspondingly.
The adjacency matrix $A(G_{FQ})$ of FQNFG involves 4 matrices they are truth, contradiction, ignorance and falsity membership values. ie)

$$A(G_{FQ}) = \left(A(T_{\mu_{FQ}}(p_ip_j)), A(C_{\mu_{FQ}}(p_ip_j)), A(I_{\mu_{FQ}}(p_ip_j)), A(F_{\mu_{FQ}}(p_ip_j))\right)$$

The "spectrum of adjacency matrix" of a FQNFG $A(G_{FQ})$ is termed as (Q, R, S, T) where Q, R, S and T represents eigen values of $A(T_{\mu_{FQ}}(p_ip_j))$, $A(C_{\mu_{FQ}}(p_ip_j))$, $A(I_{\mu_{FQ}}(p_ip_j))$ and $A(F_{\mu_{FQ}}(p_ip_j))$ respectively. **Example 1:** Consider a Graph $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$, where $\sigma_{FQ} = \{p_1, p_2, p_3, p_4\}$ and $\mu_{FQ} = \{p_1p_2, p_2p_3, p_3p_4, p_4p_1, p_1p_3\}$.

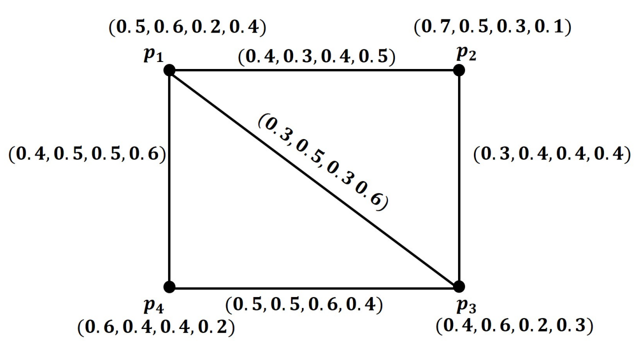

FIGURE 2. Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

σ_{FQ}	p_1	p_2	p_3	p_4
$T_{\sigma_{FQ}}$	0.5	0.7	0.4	0.6
$C_{\sigma_{FQ}}$	0.6	0.5	0.6	0.4
$I_{\sigma_{FQ}}$	0.2	0.3	0.2	0.4
$F_{\sigma_{FQ}}$	0.4	0.1	0.3	0.2

μ_{FQ}	p_1p_2	p_2p_3	p_3p_4	p_4p_1	p_1p_3
$T_{\mu_{FQ}}$	0.4	0.3	0.5	0.4	0.3
$C_{\mu_{FQ}}$	0.3	0.4	0.5	0.5	0.5
$I_{\mu_{FQ}}$	0.4	0.4	0.6	0.5	0.3
$F_{\mu_{FQ}}$	0.5	0.4	0.4	0.6	0.6

The adjacency matrix of a FQNFG is given by

$$\begin{pmatrix} (0.0,0.0,0.0,0.0) & (0.4,0.3,0.4,0.5) & (0.3,0.5,0.3,0.6) & (0.4,0.5,0.5,0.6) \\ (0.4,0.3,0.4,0.5) & (0.0,0.0,0.0,0.0) & (0.3,0.4,0.4,0.4) & (0.0,0.0,0.0,0.0) \\ (0.3,0.5,0.3,0.6) & (0.3,0.4,0.4,0.4) & (0.0,0.0,0.0,0.0) & (0.5,0.5,0.6,0.4) \\ (0.4,0.5,0.5,0.6) & (0.0,0.0,0.0,0.0) & (0.5,0.5,0.6,0.4) & (0.0,0.0,0.0,0.0) \end{pmatrix}$$

From Figure 2, Spectrum of FQNFG is
$spec(T_{\mu_{FQ}}(p_ip_j)) = \{0.9701, -0.6703, -0.3296, 0.0299\}$
$spec(C_{\mu_{FQ}}(p_ip_j)) = \{1.1490, -0.6556, -0.5, 0.0066\}$
$spec(I_{\mu_{FQ}}(p_ip_j)) = \{1.1244, -0.8270, -0.3031, 0.0057\}$
$spec(F_{\mu_{FQ}}(p_ip_j)) = \{1.3018, -0.7830, -0.5218, 0.003\}$
$spec(G_{\mu_{FQ}}(p_ip_j)) = \{(0.9701, 1.1490, 1.1244, 1.3018),$
$(-0.6703, -0.6556, -0.8270, -0.7830), (-0.3296, -0.5, -0.3031, -0.5218),$
$(0.0299, 0.0066, 0.0057, 0.003)\}$ The energy of a FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is defined as

$$\begin{aligned} E(G_{FQ}) &= \big(E(T_{\mu_{FQ}}(p_ip_j)), E(C_{\mu_{FQ}}(p_ip_j)), E(I_{\mu_{FQ}}(p_ip_j)), E(F_{\mu_{FQ}}(p_ip_j))\big) \\ &= \left(\sum_{\substack{i=1\\ \alpha_i\in Q}}^{n} |\alpha_i|, \sum_{\substack{i=1\\ \beta_i\in R}}^{n} |\beta_i|, \sum_{\substack{i=1\\ \lambda_i\in S}}^{n} |\lambda_i|, \sum_{\substack{i=1\\ \eta_i\in T}}^{n} |\eta_i|\right) \end{aligned}$$

Equienergetic FQNFGs are defined as those having equal number of vertices and identical energy values.

Theorem 3.1. Consider a FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ and adjaceny matrix $A(G_{FQ})$. If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ represent eigen values of $A(T_{\mu_{FQ}}(p_ip_j))$, $A(C_{\mu_{FQ}}(p_ip_j))$, $A(I_{\mu_{FQ}}(p_ip_j))$ and $A(F_{\mu_{FQ}}(p_ip_j))$ respectively. Then,

(1) $\sum\limits_{\substack{i=1\\ \alpha\in Q}}^{n} \alpha_i = 0,\ \sum\limits_{\substack{i=1\\ \beta\in R}}^{n} \beta_i = 0,\ \sum\limits_{\substack{i=1\\ \lambda\in S}}^{n} \lambda_i = 0,\ \sum\limits_{\substack{i=1\\ \eta\in T}}^{n} \eta_i = 0$

(2) $\sum\limits_{\substack{i=1\\ \alpha_i\in Q}}^{n} \alpha_i^2 = 2\sum\limits_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2,\ \sum\limits_{\substack{i=1\\ \beta_i\in R}}^{n} \beta_i^2 = 2\sum\limits_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)^2,$
$\sum\limits_{\substack{i=1\\ \lambda_i\in S}}^{n} \lambda_i^2 = 2\sum\limits_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)^2,\ \sum\limits_{\substack{i=1\\ \eta_i\in S}}^{n} \eta_i^2 = 2\sum\limits_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)^2.$

Proof: **(1)** W.k.t. $A(G_{FQ})$ denotes a symmetric matrix with a trace of zero (its eigen values sum to zero).

$$\text{i.e.)} \sum_{\substack{i=1\\ \alpha_i\in Q}}^{n} \alpha_i = 0$$

Similarly we get for contradiction, ignorance and false membership.
(2) To prove: $\sum\limits_{\substack{i=1\\ \alpha_i\in Q}}^{n} \alpha_i^2 = 2\sum\limits_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2$
Based on a Matrix Property,

$$tr((A(T_{\mu_{FQ}}(p_ip_j)))^2) = \sum_{\substack{i=1\\ \alpha_i\in Q}}^{n} \alpha_i^2$$

where

$$\begin{aligned} tr((A(T_{\mu_{FQ}}(p_ip_j)))^2) &= (0 + T^2_{\mu_{FQ}}(p_1p_2)) + \cdots + T^2_{\mu_{FQ}}(p_1p_n)) + \\ &\quad (T^2_{\mu_{FQ}}(p_2p_1)) + 0 + \cdots + T^2_{\mu_{FQ}}(p_2p_n)) + \cdots \\ &\quad +(T^2_{\mu_{FQ}}(p_np_1)) + T^2_{\mu_{FQ}}(p_np_2) + \cdots + 0) \\ &= 2 \sum_{1 \le i \le j \le n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 \end{aligned}$$

Hence,

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{n} \alpha_i^2 = 2 \sum_{1 \le i \le j \le n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2$$

Likewise, it follows that

$$\sum_{\substack{i=1 \\ \beta_i \in R}}^{n} \beta_i^2 = 2 \sum_{1 \le i \le j \le n} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2, \quad \sum_{\substack{i=1 \\ \lambda_i \in S}}^{n} \lambda_i^2 = 2 \sum_{1 \le i \le j \le n} \left(I_{\mu_{FQ}}(p_ip_j)\right)^2, \quad \sum_{\substack{i=1 \\ \eta_i \in T}}^{n} \eta_i^2 = 2 \sum_{1 \le i \le j \le n} \left(F_{\mu_{FQ}}(p_ip_j)\right)^2. \quad \square$$

Example 2:
From Figure 2, Calculating the graph's energy yields:
$E(T_{\mu_{FQ}}(p_ip_j) = 1.9999,\ E(C_{\mu_{FQ}}(p_ip_j) = 2.3112,\quad E(I_{\mu_{FQ}}(p_ip_j) = 2.2602,\quad E(F_{\mu_{FQ}}(p_ip_j) = 2.6096$
So, $E(G_{FQ}) = (1.9999, 2.3112, 2.2602, 2.6096)$
Also, we find the following values.

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{4} \alpha_i = 0.9701 - 0.6703 - 0.3296 + 0.0299 = 0$$

$$\sum_{\substack{i=1 \\ \beta_i \in R}}^{4} \beta_i = 1.1490 - 0.6556 - 0.5 + 0.0066 = 0$$

$$\sum_{\substack{i=1 \\ \lambda_i \in S}}^{4} \lambda_i = 1.1244 - 0.8270 - 0.3031 + 0.0057 = 0$$

$$\sum_{\substack{i=1 \\ \eta_i \in T}}^{4} \eta_i = 1.3018 - 0.7830 - 0.5218 + 0.003 = 0$$

Then

$$\sum_{\substack{i=1 \\ \alpha_i \in Q}}^{4} \alpha_i^2 = 1.5 = 2(0.75) = 2 \sum_{1 \le i \le j \le 4} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2$$

$$\sum_{\substack{i=1 \\ \beta_i \in R}}^{4} \beta_i^2 = 2 = 2(1) = 2 \sum_{1 \le i \le j \le 4} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2$$

$$\sum_{\substack{i=1 \\ \lambda_i \in S}}^{4} \lambda_i^2 = 2.04 = 2(1.02) = 2 \sum_{1 \le i \le j \le 4} \left(I_{\mu_{FQ}}(p_ip_j)\right)^2$$

$$\sum_{\substack{i=1 \\ \eta_i \in T}}^{4} \eta_i^2 = 2.58 = 2(1.29) = 2 \sum_{1 \le i \le j \le 4} \left(F_{\mu_{FQ}}(p_ip_j)\right)^2$$

Theorem 3.2. Consider a FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ with n n vertices and adjacency matrix $A(G_{FQ}) = (A(T_{\mu_{FQ}}(p_ip_j)),\ A(C_{\mu_{FQ}}(p_ip_j)),\ A(I_{\mu_{FQ}}(p_ip_j)),\ A(F_{\mu_{FQ}}(p_ip_j)))$. Then,

(1) $\sqrt{2 \sum_{1 \le i \le j \le n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + n(n-1)|T|^{\frac{2}{n}}} \le E(T_{\mu_{FQ}}(p_ip_j)) \le \sqrt{2n \sum_{1 \le i \le j \le n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2}$

(2) $\sqrt{2 \sum_{1 \le i \le j \le n} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + n(n-1)|C|^{\frac{2}{n}}} \le E(C_{\mu_{FQ}}(p_ip_j)) \le \sqrt{2n \sum_{1 \le i \le j \le n} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2}$

(3) $\sqrt{2\sum\limits_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2+n(n-1)|I|^{\frac{2}{n}}}\leq E(I_{\mu_{FQ}}(p_ip_j))\leq\sqrt{2n\sum\limits_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2}$

(4) $\sqrt{2\sum\limits_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2+n(n-1)|F|^{\frac{2}{n}}}\leq E(F_{\mu_{FQ}}(p_ip_j))\leq\sqrt{2n\sum\limits_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2}$

where $|T|$, $|C|$, $|I|$ and $|F|$ represent determinant of $A(T_{\mu_{FQ}}(p_ip_j))$, $A(C_{\mu_{FQ}}(p_ip_j))$, $A(I_{\mu_{FQ}}(p_ip_j))$, and $A(F_{\mu_{FQ}}(p_ip_j))$ respectively.

Proof: **(1) Upper Bound (UB)**
Utilizing Cauchy-Schwarz inequality to the n numbers $1, 1, \cdots 1$ and $|\lambda_1|, |\lambda_2|, \cdots |\lambda_n|$, we get

$$\sum_{i=1}^{n}|\lambda_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{1}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2+2\sum_{1\leq i\leq j\leq n}\lambda_i\lambda_j \tag{2}$$

Equating coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^{n}(\lambda-\lambda_i)=|A(G_{FQ})-\lambda I|$$

then

$$\sum_{1\leq i\leq j\leq n}\lambda_i\lambda_j=-\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2 \tag{3}$$

Substituting (3) into (2) gives

$$\sum_{i=1}^{n}|\lambda_i|^2=2\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2 \tag{4}$$

Substituting (4) into (2) yields

$$\begin{aligned}\sum_{i=1}^{n}|\lambda_i|^2 &\leq \sqrt{n}\sqrt{2\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2}=\sqrt{2n\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2}\\ E(T_{\mu_{FQ}}(p_ip_j)) &\leq \sqrt{2n\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2}\end{aligned}$$

(2) Lower bound (LB)

$$\begin{aligned}\left(E(T_{\mu_{FQ}}(p_ip_j))\right)^2 &= \left(\sum_{i=1}^{n}|\lambda_i|^2\right)^2\\ &= \sum_{i=1}^{n}|\lambda_i|^2+2\sum_{1\leq i\leq j\leq n}|\lambda_i\lambda_j|\\ &= 2\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2+\frac{2n(n-1)}{2}AM\{|\lambda_i\lambda_j|\}\end{aligned}$$

W.k.t. $AM\{|\lambda_i\lambda_j|\}\geq GM\{|\lambda_i\lambda_j|\}, 1\leq i\leq j\leq n$
$\left(E(T_{\mu_{FQ}}(p_ip_j))\right)\geq\sqrt{2\sum\limits_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2+n(n-1)GM\{|\lambda_i\lambda_j|\}}$

and then

$$\begin{aligned} GM\{|\lambda_i\lambda_j|\} &= \left(\prod_{1\leq i\leq j\leq n} |\lambda_i\lambda_j|\right)^{\dfrac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n} |\lambda_i|^{n-1}\right)^{\dfrac{2}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\dfrac{2}{n}} \\ &= |T|^{\dfrac{2}{n}} \end{aligned}$$

$$\big(E(T_{\mu_{FQ}}(p_ip_j))\big) \geq \sqrt{2\sum_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2 + n(n-1)|T|^{\dfrac{2}{n}}}$$

$$\begin{aligned} \therefore \sqrt{2\sum_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2 + n(n-1)|T|^{\dfrac{2}{n}}} &\leq \big(E(T_{\mu_{FQ}}(p_ip_j))\big) \\ &\leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2} \end{aligned}$$

Correspondingly, it follows that

$$\begin{aligned} \sqrt{2\sum_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)^2 + n(n-1)|C|^{\dfrac{2}{n}}} &\leq \big(E(C_{\mu_{FQ}}(p_ip_j))\big) \\ &\leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)^2} \end{aligned}$$

$$\begin{aligned} \sqrt{2\sum_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)^2 + n(n-1)|I|^{\dfrac{2}{n}}} &\leq \big(E(I_{\mu_{FQ}}(p_ip_j))\big) \\ &\leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)^2} \end{aligned}$$

$$\begin{aligned} \sqrt{2\sum_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)^2 + n(n-1)|F|^{\dfrac{2}{n}}} &\leq \big(E(F_{\mu_{FQ}}(p_ip_j))\big) \\ &\leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)^2} \end{aligned}$$

□

Example 3: From Figure 2, Let's calculate LB and UB for FQNFG.
$\big(E(T_{\mu_{FQ}}(p_ip_j))\big) = 1.9999$, LB = 1.5684 and UB = 2.4495.

$$\therefore 1.5684 \leq 1.9999 \leq 2.4495$$

$\big(E(C_{\mu_{FQ}}(p_ip_j))\big) = 2.3112$, LB $= 1.6125$ and UB $= 2.8284$.

$$\therefore 1.6125 \leq 2.3112 \leq 2.8284$$

$\big(E(I_{\mu_{FQ}}(p_ip_j))\big) = 2.2602$, LB $= 1.5875$ and UB $= 2.8566$.

$$\therefore 1.5875 \leq 2.2602 \leq 2.8566$$

$\big(E(F_{\mu_{FQ}}(p_ip_j))\big) = 2.6096$, LB $= 1.7493$ and UB $= 3.2125$.

$$\therefore 1.7493 \leq 2.6096 \leq 3.2125$$

4 | Laplacian Energy of Fermatean Quadripartitioned Neutro-sophic Fuzzy Graph

Consider $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is a FQNFG with n vertices. Then degree matrix $D(G_{FQ}) = (D(T_{\mu_{FQ}}(p_ip_j)), D(C_{\mu_{FQ}}(p_ip_j)), D(I_{\mu_{FQ}}(p_ip_j)), D(F_{\mu_{FQ}}(p_ip_j))) = [d_{ij}]$, of G_{FQ} is a $n \times n$ diagonal matrix termed as

$$d_{ij} = \begin{cases} d_{G_{FQ}}(p_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The Laplacian matrix (LM) of a FQNFG $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is termed as $L(G_{FQ}) = (L(T_{\mu_{FQ}}(p_ip_j)), L(C_{\mu_{FQ}}(p_ip_j)), L(I_{\mu_{FQ}}(p_ip_j)), L(F_{\mu_{FQ}}(p_ip_j))) = D(G_{FQ}) - A(G_{FQ})$, where $A(G_{FQ})$ and $D(G_{FQ})$ represent an adjacency matrix and degree matrix of FQNFG. The spectrum of LM of FQNFG, $L(G_{FQ})$ is denoted by (Q_L, R_L, S_L, T_L), where Q_L, R_L, S_L and T_L are sets of Laplacian eigen values of $L(T_{\mu_{FQ}}(p_ip_j))$, $L(C_{\mu_{FQ}}(p_ip_j))$, $L(I_{\mu_{FQ}}(p_ip_j))$ and $L(F_{\mu_{FQ}}(p_ip_j))$ respectively. **Example 4:**
Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a FQNFG, where $\sigma_{FQ} = (p_1, p_2, p_3, p_4)$ and $\mu_{FQ} = (p_1p_2, p_2p_3, p_3p_1, p_3p_4, p_2p_4)$

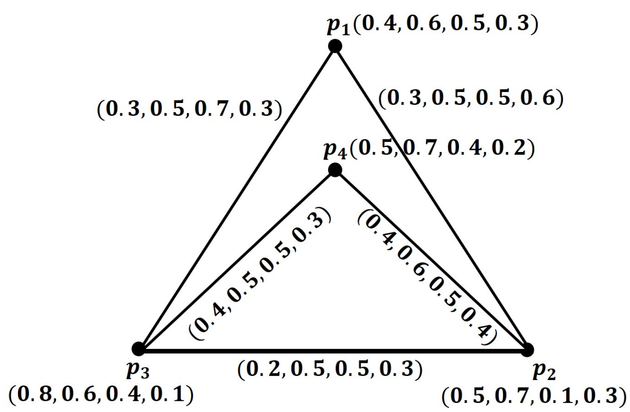

FIGURE 3. Fermatean Quadripartitioned Neutrosophic Fuzzy Graph

σ_{FQ}	p_1	p_2	p_3	p_4
$T_{\sigma_{FQ}}$	0.4	0.5	0.8	0.5
$C_{\sigma_{FQ}}$	0.6	0.7	0.6	0.7
$I_{\sigma_{FQ}}$	0.5	0.1	0.4	0.4
$F_{\sigma_{FQ}}$	0.3	0.3	0.1	0.2

The adjacency matrix of a FQNFG is given by

$$\begin{pmatrix} (0.0, 0.0, 0.0, 0.0) & (0.3, 0.5, 0.5, 0.6) & (0.3, 0.5, 0.7, 0.3) & (0.0, 0.0, 0.0, 0.0) \\ (0.3, 0.5, 0.5, 0.6) & (0.0, 0.0, 0.0, 0.0) & (0.2, 0.5, 0.5, 0.3) & (0.4, 0.6, 0.5, 0.4) \\ (0.3, 0.5, 0.7, 0.3) & (0.2, 0.5, 0.5, 0.3) & (0.0, 0.0, 0.0, 0.0) & (0.4, 0.5, 0.5, 0.3) \\ (0.0, 0.0, 0.0, 0.0) & (0.4, 0.6, 0.5, 0.4) & (0.4, 0.5, 0.5, 0.3) & (0.0, 0.0, 0.0, 0.0) \end{pmatrix}$$

μ_{FQ}	p_1p_2	p_2p_3	p_3p_1	p_3p_4	p_2p_4
$T_{\mu_{FQ}}$	0.3	0.2	0.3	0.4	0.4
$C_{\mu_{FQ}}$	0.5	0.5	0.5	0.5	0.6
$I_{\mu_{FQ}}$	0.5	0.5	0.7	0.5	0.5
$F_{\mu_{FQ}}$	0.6	0.4	0.4	0.3	0.4

The Laplacian matrix of a FQNFG is given by

$$L(G_{FQ}) = \begin{pmatrix} (0.6, 1.0, 1.2, 0.9) & (-0.3, -0.5, -0.5, -0.6) & (-0.3, -0.5, -0.7, -0.4) & (0.0, 0.0, 0.0, 0.0) \\ (-0.3, -0.5, -0.5, -0.6) & (1.1, 1.6, 1.5, 1.3) & (-0.2, -0.5, -0.5, -0.4) & (-0.4, -0.6, -0.5, -0.4) \\ (-0.3, -0.5, -0.7, -0.4) & (-0.2, -0.5, -0.5, -0.4) & (0.9, 1.5, 1.7, 0.9) & (-0.4, -0.5, -0.5, -0.3) \\ (0.0, 0.0, 0.0, 0.0) & (-0.4, -0.6, -0.5, -0.4) & (-0.4, -0.5, -0.5, -0.3) & (0.8, 1.1, 1.0, 0.7) \end{pmatrix}$$

The Laplacian spectrum of a FQNFG is
Laplacian $spec(T_{\mu_{FQ}}(p_ip_j)) = \{0.0442, 0.6740, 1.4952, 1.1866\}$
Laplacian $spec(C_{\mu_{FQ}}(p_ip_j)) = \{0, 1.0432, 2.1568, 2\}$
Laplacian $spec(I_{\mu_{FQ}}(p_ip_j)) = \{0, 1.0755, 2.3245, 2\}$
Laplacian $spec(F_{\mu_{FQ}}(p_ip_j)) = \{-0.1044, 0.7650, 1.8304, 1.3090\}$
Laplacian $spec(G_{FQ}(p_ip_j)) = \{(0.0442, 0, 0, -0.1044), (0.6740, 1.0432, 1.0755, 0.7650), (1.4952, 2.1568, 2.3245, 1.8304), (1.1866, 2, 2, 1.3090)\}$

Theorem 4.3. Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a FQNFG and $L(G_{FQ}) = (L(T_{\mu_{FQ}}(p_ip_j)),\ L(C_{\mu_{FQ}}(p_ip_j)),\ L(I_{\mu_{FQ}}(p_ip_j)),\ L(F_{\mu_{FQ}}(p_ip_j)))$ be the LM of G_{FQ}. If $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_n$, $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$, and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ were eigen values of $L(T_{\mu_{FQ}}(p_ip_j))$, $L(C_{\mu_{FQ}}(p_ip_j))$, $L(I_{\mu_{FQ}}(p_ip_j))$ and $L(F_{\mu_{FQ}}(p_ip_j))$ correspondingly. Therefore,

(1) $\sum_{\substack{i=1\\ \gamma_i \in Q_L}}^{n} \gamma_i = 2\sum_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)$, $\sum_{\substack{i=1\\ \phi_i \in R_L}}^{n} \phi_i = 2\sum_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)$, $\sum_{\substack{i=1\\ \tau_i \in S_L}}^{n} \tau_i = 2\sum_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)$, $\sum_{\substack{i=1\\ \rho_i \in T_L}}^{n} \rho_i = 2\sum_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)$

(2) $\sum_{\substack{i=1\\ \gamma_i \in Q_L}}^{n} \gamma_i^2 = 2\sum_{1\leq i\leq j\leq n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_i)$,
$\sum_{\substack{i=1\\ \phi_i \in R_L}}^{n} \phi_i^2 = 2\sum_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{C_{\mu_{FQ}}(p_ip_j)}(p_i)$,
$\sum_{\substack{i=1\\ \tau_i \in S_L}}^{n} \tau_i^2 = 2\sum_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{I_{\mu_{FQ}}(p_ip_j)}(p_i)$,
$\sum_{\substack{i=1\\ \rho_i \in T_L}}^{n} \rho_i^2 = 2\sum_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{F_{\mu_{FQ}}(p_ip_j)}(p_i)$.

Proof: **(1)** We know that $L(G_{FQ})$ denotes a symmetric matrix of positive values. Then,

$$\begin{aligned} \sum_{\substack{i=1\\ \gamma_i \in Q_L}}^{n} \gamma_i &= tr(L(G_{FQ})) \\ &= \sum_{i=1}^{n} d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) \\ &= 2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j) \end{aligned}$$

Likewise, it follows that,
$\sum_{\substack{i=1\\ \phi_i \in R_L}}^{n} \phi_i = 2\sum_{1\leq i\leq j\leq n} \big(C_{\mu_{FQ}}(p_ip_j)\big)$, $\sum_{\substack{i=1\\ \tau_i \in S_L}}^{n} \tau_i = 2\sum_{1\leq i\leq j\leq n} \big(I_{\mu_{FQ}}(p_ip_j)\big)$, $\sum_{\substack{i=1\\ \rho_i \in T_L}}^{n} \rho_i = 2\sum_{1\leq i\leq j\leq n} \big(F_{\mu_{FQ}}(p_ip_j)\big)$

(2) By the definition of LM,

$$L(T_{\mu_{FQ}}(p_ip_j)) = \begin{pmatrix} d_{T_{\mu_{FQ}}(p_ip_j)}(p_1) & -T_{\mu_{FQ}}(p_1p_2) & \cdots & -T_{\mu_{FQ}}(p_1p_n) \\ -T_{\mu_{FQ}}(p_2p_1) & d_{T_{\mu_{FQ}}(p_ip_j)}(p_2) & \cdots & -T_{\mu_{FQ}}(p_2p_n) \\ \vdots & \vdots & \ddots & \vdots \\ -T_{\mu_{FQ}}(p_np_1) & -T_{\mu_{FQ}}(p_np_2) & \cdots & d_{T_{\mu_{FQ}}(p_ip_j)}(p_n) \end{pmatrix}$$ By matrix trace theory,

$$tr((L(T_{\mu_{FQ}}(p_ip_j)))^2) = \sum_{\substack{i=1 \\ \gamma_i \in Q_L}}^{n} \gamma_i^2$$

where

$$\begin{aligned} tr((L(T_{\mu_{FQ}}(p_ip_j)))^2) &= \Big(d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_1) + T^2_{\mu_{FQ}}(p_1p_2) + \cdots + T^2_{\mu_{FQ}}(p_1p_n)\Big) + \\ &\quad \Big(T^2_{\mu_{FQ}}(p_2p_1) + d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_2) + \cdots + T^2_{\mu_{FQ}}(p_2p_n)\Big) + \cdots \\ &\quad + \Big(T^2_{\mu_{FQ}}(p_np_1) + T^2_{\mu_{FQ}}(p_np_2) + \cdots + d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_n)\Big) \\ &= 2\sum_{1\le i\le j\le n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_i) \\ \therefore \sum_{\substack{i=1 \\ \gamma_i \in Q_L}}^{n} \gamma_i^2 &= 2\sum_{1\le i\le j\le n} \big(T_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{T_{\mu_{FQ}}(p_ip_j)}(p_i) \end{aligned}$$

Likewise, we get

$\sum_{\substack{i=1 \\ \phi_i \in R_L}}^{n} \phi_i^2 = 2\sum_{1\le i\le j\le n} \big(C_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{C_{\mu_{FQ}}(p_ip_j)}(p_i)$,

$\sum_{\substack{i=1 \\ \tau_i \in S_L}}^{n} \tau_i^2 = 2\sum_{1\le i\le j\le n} \big(I_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{I_{\mu_{FQ}}(p_ip_j)}(p_i)$,

$\sum_{\substack{i=1 \\ \rho_i \in T_L}}^{n} \rho_i^2 = 2\sum_{1\le i\le j\le n} \big(F_{\mu_{FQ}}(p_ip_j)\big)^2 + \sum_{i=1}^{n} d^2_{F_{\mu_{FQ}}(p_ip_j)}(p_i)$. □

The LE of FQNFG $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ is denoted by

$$\begin{aligned} LE(G_{FQ}) &= (LE(T_{\mu_{FQ}}(p_ip_j)), LE(C_{\mu_{FQ}}(p_ip_j)), LE(I_{\mu_{FQ}}(p_ip_j)), LE(F_{\mu_{FQ}}(p_ip_j))) \\ &= \left(\sum_{i=1}^{n}|\delta_i|, \sum_{i=1}^{n}|\Delta_i|, \sum_{i=1}^{n}|\theta_i|, \sum_{i=1}^{n}|\zeta_i|\right) \end{aligned}$$

where,

$$\delta_i = \gamma_i - \frac{2\sum_{1\le i\le j\le n} T_{\mu_{FQ}}(p_ip_j)}{n}; \quad \Delta_i = \phi_i - \frac{2\sum_{1\le i\le j\le n} C_{\mu_{FQ}}(p_ip_j)}{n};$$

$$\theta_i = \tau_i - \frac{2\sum_{1\le i\le j\le n} I_{\mu_{FQ}}(p_ip_j)}{n}; \quad \zeta_i = \rho_i - \frac{2\sum_{1\le i\le j\le n} F_{\mu_{FQ}}(p_ip_j)}{n}.$$

Theorem 4.4. Let $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ be a FQNFG and $L(G_{FQ})$ be the LM of G_{FQ}. If $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n$, $\phi_1 \ge \phi_2 \ge \cdots \ge \phi_n$, $\tau_1 \ge \tau_2 \ge \cdots \ge \tau_n$, and $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n$ were eigen values of $L(T_{\mu_{FQ}}(p_ip_j))$, $L(C_{\mu_{FQ}}(p_ip_j))$, $L(I_{\mu_{FQ}}(p_ip_j))$ and $L(F_{\mu_{FQ}}(p_ip_j))$ respectively and $\delta_i = \gamma_i - \frac{2\sum_{1\le i\le j\le n} T_{\mu_{FQ}}(p_ip_j)}{n}$; $\Delta_i = \phi_i - \frac{2\sum_{1\le i\le j\le n} C_{\mu_{FQ}}(p_ip_j)}{n}$; $\theta_i = \tau_i - \frac{2\sum_{1\le i\le j\le n} I_{\mu_{FQ}}(p_ip_j)}{n}$; $\zeta_i = \rho_i - \frac{2\sum_{1\le i\le j\le n} F_{\mu_{FQ}}(p_ip_j)}{n}$. Then $\sum_{i=1}^{n}\delta_i = 0$, $\sum_{i=1}^{n}\Delta_i = 0$, $\sum_{i=1}^{n}\theta_i = 0$, $\sum_{i=1}^{n}\zeta_i = 0$, $\sum_{i=1}^{n}\delta_i^2 = 2\mathbb{N}_T$, $\sum_{i=1}^{n}\Delta_i^2 = 2\mathbb{N}_C$, $\sum_{i=1}^{n}\theta_i^2 = 2\mathbb{N}_I$, $\sum_{i=1}^{n}\zeta_i^2 = 2\mathbb{N}_F$, where

$$\mathbb{N}_T = \sum_{1\leq i\leq j\leq n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

$$\mathbb{N}_C = \sum_{1\leq i\leq j\leq n} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

$$\mathbb{N}_I = \sum_{1\leq i\leq j\leq n} \left(I_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

$$\mathbb{N}_F = \sum_{1\leq i\leq j\leq n} \left(F_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

Theorem 4.5. Consider FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ with n vertices and $L(G_{FQ}) = (L(T_{\mu_{FQ}}(p_ip_j)), L(C_{\mu_{FQ}}(p_ip_j)), L(I_{\mu_{FQ}}(p_ip_j)), L(F_{\mu_{FQ}}(p_ip_j)))$ be the LM of G_{FQ}. Hence,

(1) $$LE(T_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(2) $$LE(C_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(3) $$LE(I_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \left(I_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(4) $$LE(F_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \left(F_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

Proof: Applying Cauchy-Schwarz inequality to the n numbers $1, 1, \cdots, 1$ and $|\delta_1|, |\delta_2|, \cdots, |\delta_n|$ results in,

$$\sum_{i=1}^{n} |\delta_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\delta_i|^2}$$

$$LE(T_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{n}\sqrt{2\mathbb{N}_T} = \sqrt{2n\mathbb{N}_T}$$

Since $\mathbb{N}_T = \sum_{1\leq i\leq j\leq n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$

$$\therefore LE(T_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n} \left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

Likewise, we get

$$LE(C_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n}\left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

$$LE(I_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

$$LE(F_{\mu_{FQ}}(p_ip_j)) \leq \sqrt{2n\sum_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2 + n\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$ □

Theorem 4.6. Consider FQNFG, $G_{FQ} = (\sigma_{FQ}, \mu_{FQ})$ with n vertices and $L(G_{FQ}) = (L(T_{\mu_{FQ}}(p_ip_j)), L(C_{\mu_{FQ}}(p_ip_j)), L(I_{\mu_{FQ}}(p_ip_j)), L(F_{\mu_{FQ}}(p_ip_j)))$ are LM of G_{FQ}. Thus,

(1) $$LE(T_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(2) $$LE(C_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(3) $$LE(I_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

(4) $$LE(F_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

Proof:

$$\left(\sum_{i=1}^{n}|\delta_i|\right)^2 = \sum_{i=1}^{n}|\delta_i|^2 + 2\sum_{1\leq i\leq j\leq n}|\delta_i\delta_j| \geq 4\mathbb{N}_T$$

$$LE(T_{\mu_{FQ}}(p_ip_j)) \geq 2\mathbb{N}_T$$

We know that

$$\mathbb{N}_T = \sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

$$\therefore LE(T_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

Similary, it follows that

$$LE(C_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(C_{\mu_{FQ}}(p_ip_j)\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n}C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

$$LE(I_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2}$$

$$LE(F_{\mu_{FQ}}(p_ip_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2} \qquad \square$$

Theorem 4.7. Consider FQNFG, $G_{FQ}=(\sigma_{FQ},\mu_{FQ})$ with n vertices and $L(G_{FQ})=(L(T_{\mu_{FQ}}(p_ip_j)),$ $L(C_{\mu_{FQ}}(p_ip_j))$, $L(I_{\mu_{FQ}}(p_ip_j))$, $L(F_{\mu_{FQ}}(p_ip_j)))$ are LM of G_{FQ}. Thus,

(1) $$LE(T_{\mu_{FQ}}(p_ip_j)) \leq |\delta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\delta_1^2\right)}$$

(2) $$LE(C_{\mu_{FQ}}(p_ip_j)) \leq |\Delta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(C_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\Delta_1^2\right)}$$

(3) $$LE(I_{\mu_{FQ}}(p_ip_j)) \leq |\theta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\theta_1^2\right)}$$

(4) $$LE(F_{\mu_{FQ}}(p_ip_j)) \leq |\zeta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\zeta_1^2\right)}$$

Proof: The Cauchy-Schwarz Inequality yields, $\sum_{i=1}^{n}|\delta_i| \leq \sqrt{n\sum_{i=1}^{n}|\delta_i|^2}$, $\sum_{i=2}^{n}|\delta_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}|\delta_i|^2}$

$$\begin{aligned} LE(T_{\mu_{FQ}}(p_ip_j))-|\delta_1| &\leq \sqrt{(n-1)(2\mathbb{N}_T-\delta_1^2)} \\ LE(T_{\mu_{FQ}}(p_ip_j)) &\leq |\delta_1|+\sqrt{(n-1)(2\mathbb{N}_T-\delta_1^2)} \end{aligned}$$

We know that,

$$\mathbb{N}_T=\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2$$

$$\therefore LE(T_{\mu_{FQ}}(p_ip_j)) \leq |\delta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(T_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{T_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} T_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\delta_1^2\right)}$$

Similarly, we can prove

$$LE(C_{\mu_{FQ}}(p_ip_j)) \leq |\Delta_1|+\sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(C_{\mu_{FQ}}(p_ip_j)\right)^2+\sum_{i=1}^{n}\left(d_{C_{\mu_{FQ}}(p_ip_j)}(p_i)-\frac{2\sum_{1\leq i\leq j\leq n} C_{\mu_{FQ}}(p_ip_j)}{n}\right)^2-\Delta_1^2\right)}$$

$$LE(I_{\mu_{FQ}}(p_ip_j)) \leq |\theta_1| + \sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(I_{\mu_{FQ}}(p_ip_j)\right)^2 + \sum_{i=1}^{n}\left(d_{I_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} I_{\mu_{FQ}}(p_ip_j)}{n}\right)^2 - \theta_1^2\right)}$$

$$LE(F_{\mu_{FQ}}(p_ip_j)) \leq |\zeta_1| + \sqrt{(n-1)\left(2\sum_{1\leq i\leq j\leq n}\left(F_{\mu_{FQ}}(p_ip_j)\right)^2 + \sum_{i=1}^{n}\left(d_{F_{\mu_{FQ}}(p_ip_j)}(p_i) - \frac{2\sum_{1\leq i\leq j\leq n} F_{\mu_{FQ}}(p_ip_j)}{n}\right)^2 - \zeta_1^2\right)}$$

□

5 | Application: The Quest for a new Laptop

To purchase a new laptop using MCDM, the criteria are

(1) Processor

(2) Memory & Storage

(3) Battery life

(4) Display quality

Embarking on the New Laptop Hunt requires careful consideration of several key criteria. Processing Power is paramount, demanding fast performance and the ability to handle seamless multitasking. Equally important are Memory & Storage, requiring ample RAM for smooth operation and sufficient storage to accommodate all essential files and programs. Battery Life is a crucial factor for uninterrupted productivity, ideally lasting for the entire workday. Finally, Display Quality plays a significant role in both work and entertainment, necessitating vibrant visuals and crisp details for an optimal viewing experience.
Now, using a decision-making method with 4 criteria, we will consider 4 laptops to find the best one to buy.
Laptop 1:

(1) Processor – (0.3, 0.4, 0.3, 0.1)

(2) Memory & Storage – (0.6, 0.5, 0.3, 0.2)

(3) Battery life – (0.3, 0.5, 0.2, 0.1)

(4) Display quality – (0.2, 0.1, 0.6, 0.3)

Laptop 2:

(1) Processor – (0.8, 0.2, 0.1, 0.1)

(2) Memory & Storage – (0.7, 0.4, 0.3, 0.2)

(3) Battery life – (0.7, 0.6, 0.3, 0.4)

(4) Display quality – (0.5, 0.8, 0.2, 0.3)

Laptop 3:

(1) Processor – (0.6, 0.6, 0.5, 0.3)

(2) Memory & Storage – (0.5, 0.4, 0.6, 0.2)

(3) Battery life – (0.8, 0.2, 0.6, 0.2)

(4) Display quality – (0.2, 0.3, 0.5, 0.3)

Laptop 4:

(1) Processor – (0.6, 0.5, 0.3, 0.1)

(2) Memory & Storage – (0.4, 0.4, 0.5, 0.1)

(3) Battery life – (0.6, 0.3, 0.1, 0.2)

(4) Display quality – (0.7, 0.6, 0.4, 0.2)

The criteria have associated weights given by the vector W = (0.6,0.4,0.3,0.2). In the decision matrix P, row denotes laptop and column represents one of the four criteria.

$$P = \begin{pmatrix} (0.3,0.4,0.3,0.1) & (0.6,0.5,0.3,0.2) & (0.3,0.5,0.2,0.1) & (0.2,0.1,0.6,0.3) \\ (0.8,0.2,0.1,0.1) & (0.7,0.4,0.3,0.2) & (0.7,0.6,0.3,0.4) & (0.5,0.8,0.2,0.3) \\ (0.6,0.6,0.5,0.3) & (0.5,0.4,0.6,0.2) & (0.8,0.2,0.6,0.2) & (0.2,0.3,0.5,0.3) \\ (0.6,0.5,0.3,0.1) & (0.4,0.4,0.5,0.1) & (0.6,0.3,0.1,0.2) & (0.7,0.6,0.4,0.2) \end{pmatrix}$$

We will compute the Fermatean Quadripartitioned Neutrosophic arithmetic average across all laptops for each criterion, resulting in a single Fermatean Quadripartitioned Neutrosophic value per criterion. The aggregated Fermatean quadripartioned neutrosophic values are as follows.

$$L_1 = \left(\frac{0.3+0.6+0.3+0.2}{4}, \frac{0.4+0.5+0.5+0.1}{4}, \frac{0.3+0.3+0.2+0.6}{4}, \frac{0.1+0.2+0.1+0.3}{4}\right)$$
$$= (0.35, 0.375, 0.35, 0.175)$$

Similarly we Calculate for other Laptops as

$L_2 = (0.675, 0.5, 0.225, 0.25)$; $L_3 = (0.525, 0.375, 0.55, 0.25)$; $L_4 = (0.575, 0.45, 0.325, 0.15)$.

Using these aggregated values for each criterion, we will now determine the overall score and accuracy for each laptop. We will use the aggregated Fermatean Quadripartitioned Neutrosophic values from the previous step to compute score and accuracy for all laptops seperately.

Score equation formula is

$$S = \frac{m}{2} \times \sum_{i=0}^{m} \left(\frac{W_T T_i + W_C C_i + W_I I_i + W_F F_i}{W_T + W_C + W_I + W_F}\right)$$

After calculating We get,

$S(L_1) = 0.67$; $S(L_2) = 0.96$; $S(L_3) = 0.9$; $S(L_4) = 0.87$

The accuracy function formula is

$$A = \sum_{i=0}^{m} \sqrt{\left(\frac{(W_T T_i)^2 + (W_C C_i)^2 + (W_I I_i)^2 + (W_F F_i)^2}{2}\right)}$$

After computation,

$A(L_1) = 0.1985$; $A(L_2) = 0.325$; $A(L_3) = 0.2751$; $A(L_4) = 0.2844$.

We can now rank the laptops based on their calculated scores and accuracy values as follows:

$L_2 > L_4 > L_3 > L_1$. In conclusion Laptop L_2 is identified as the best Laptop for purchasing.

6 | Conclusion

Fermatean Quadripartitioned Neutrosophic Fuzzy Graph (FQNFG) provides a suitable modelling framework. These models generally offer improved precision, greater flexibility and enhanced compatibility compared to other fuzzy approaches. This manuscript dealt with the new idea of Energy and LE of FQNFG . The LB and UB of Energy and LE for FQNFG were derived. The energy of FQNFG holds potential for applications in Computer Science, Chemistry, Risk Assessment, Medical diagnosis etc. The proposed methodology offers the most effective method for laptop acquisition. The Future work will extend to energy of FQNFGs to Double Layered Fuzzy Graph, Hesitant Fuzzy Graph, Spherical Fuzzy Graph etc.

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