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Fuzzy Computational Procedure for System Length and Waiting Time Distribution in an M^X/D/C/N Queue under Parameter Uncertainty

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Abstract

This paper introduces a fuzzy computational method for analyzing a finite buffer multiple server $M^X/D/c/N$ queuing system with uncertain variables. Unlike common models that assume precise knowledge of arrival, service, and batch size distributions, this model handles these variables as triangular fuzzy numbers. This reflects parameter uncertainty caused by limited data and operational changes. The fuzzy characteristic equation is created, and using the α-cut method, the problem changes into a set of interval polynomial equations. We show that fuzzy stationary probabilities do exist for the system. Using interval arithmetic and the Laplace transform with Padé approximation, we find solutions for fuzzy mean queue length, arrival rate, and mean waiting time. We then use a practical example based on public services to show how the method works. A sensitivity study measures how uncertainty spreads into congestion levels. The results show that parameter uncertainty grows when systems approach saturation, requiring strong capacity planning. This approach maintains the analytical style of the root-based method while giving specific performance limits, making it a mathematically sound and understandable extension of multi-server queuing theory under uncertainty.

Keywords: Fuzzy queueing system, Batch arrival, Finite buffer, Characteristic roots, α cut method, Waiting time distribution.

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1 | Introduction

Queueing models are a cornerstone of operations research and applied probability theories to capture delay dynamics in service systems. Classical models such as the $M^X/D/c/N$ queue consider the arrival rate, service time and the batch size distribution as known precisely. These assumptions allow explicit derivations of stationary distributions and performance measures by powerful analytic tools such as generating functions and roots of characteristic equations.

Nonetheless, this assumption of parameter exactness is untenable in many real-life situations. Often arrival rates are inferred from prior limited observations and can vary as a function of time of day, season, or other environmental factors. Service duration is subject to case complexity and operator skill, which questions deterministic specification, especially in public service systems, hospitals, and document processing plants. Batch sizes may be influenced by social or organizational behavior and may not be characterized by stable long run frequencies. In these contexts, the unpredictability stems both from the fundamental stochastic nature of the processes and from the lack of definitive knowledge about the values of the parameters. This uncertainty-type is epistemic instead of purely probabilistic.

Classical probability theory takes care of randomness, but it is not sufficient to describe lack of sharpness or ambiguity in the knowledge of parameters. Point estimates and classical confidence intervals may not adequately reflect the extent of information imprecision in cases where sample sizes are small or expert judgment has been used to inform the estimation. Fuzzy set theory, initiated by Zadeh, has been successfully used to model such imprecision by employing membership functions as opposed to crisp parameter values. By modeling arrival rate, service time, and batch size as fuzzy numbers, one can propagate parameter ambiguity through the analytical structure of the queueing model and obtain performance measures in the form of uncertainty bands rather than single deterministic outputs.

Despite extensive research on fuzzy queueing systems, most existing studies focus on Markovian single-server models or rely on approximate numerical methods. The rigorous integration of fuzzy parameterization with root-based analytical methods for finite buffer multi-server queues remains limited. The $M^X/D/c/N$ model is particularly suitable for such investigation because its classical analysis is heavily dependent on the structure of the characteristic equation and its roots within the unit disk. Extending this analytical machinery into a fuzzy framework requires careful treatment of interval polynomials, root preservation, and reconstruction of fuzzy stationary probabilities.

Therefore, the primary motivation of this study is to develop a mathematically rigorous and computationally tractable fuzzy extension of the $M^X/D/c/N$ queue that preserves the structural elegance of the classical roots-based method while incorporating epistemic uncertainty in key parameters. Such an extension provides not only expected performance measures but also explicit uncertainty bounds, thereby offering enhanced decision support for capacity planning and congestion management in real-world service systems.

Recent research in fuzzy queueing theory has focused on extending classical queueing models to incorporate uncertainty in key parameters such as arrival rates, service times, and network dynamics. A class of studies employs the α-cut approach and Zadeh's extension principle to transform fuzzy queues into families of crisp queues, thereby facilitating analytical computation of membership functions for performance measures under uncertainty. For example, Chen analyzed strategic customer behaviour in fuzzy Markovian queues where system parameters are fuzzy numbers and derived membership functions for optimal strategies and performance metrics using parametric nonlinear programming methods [7]. Similarly, Liu introduced an uncertain queueing model with group arrivals that accounts for the belief degree in customer behaviour and evaluates performance under interval uncertainty models [12]. Bulk arrivals in queues with fuzzy service and arrival rates have been studied through α-cut based MINLP formulations to obtain the performance measure membership functions [15]. Multi-server fuzzy queueing system under reneging customers and finite capacity has been analyzed, where recursive and analytical approaches to determine fuzzy queue length and waiting-time as interval-valued measures are presented [17]. Further, a fuzzy mathematical modelling of a multi-server queuing system with uncertainty in the number of servers, arrival rates and service rates for decision support in dynamic environment is introduced [18]. Together they represent the types of pursuits that showcase the diversity of modern fuzzy queueing research and indicate strong motivation to develop analytically sound frameworks for such problems that maintain derived structural properties with uncertainty.

Recent analytical developments in finite buffer and multi-server queueing systems continue to rely on generating function methods and root localization techniques. The matrix-analytic framework introduced by Neuts [16] provided a structured approach for analyzing multi-server and bulk arrival queues through characteristic roots and structured Markov chains. Finite capacity multi-server systems have also been studied using analytic continuation and probability generating function factorization, preserving root structures inside the unit disk to determine stationary probabilities. Medhi's comprehensive treatment of stochastic models [14] and Kleinrock's foundational work on queueing systems [10] provide rigorous exposition of such analytic methods for multi-server queues. Recently developed computational methods have focused on the algorithmic determination of the steady-state probabilities through root-finding numerical procedures for multi-server systems with finite capacity, which demonstrates that classical characteristic equation methods still play an essential role in structured queues. However, these deterministic formulations assume precise parameter knowledge and do not directly incorporate epistemic uncertainty. Consequently, while the analytical backbone of the $M^X/D/c/N$ model is well established, its integration with fuzzy parameterization remains largely unexplored in rigorous roots-based form.

After deterministic queueing theory, the theoretical level for modeling epistemic errors for uncertain variables is achieved with fuzzy set theory and interval analysis. Zadeh's seminal paper on fuzzy sets [21] based imprecision on membership representation, which was formalized by α-cut representations and extension principles leading to transfer of fuzzy functions into families of interval-valued problems. The theoretical background of fuzzy numbers, their arithmetic as well as α-cut has been thoroughly and nicely presented in [8] and [22] which This accounts for a very powerful methodology for performing uncertainty propagation across nonlinear functions. In parallel, interval analysis and perturbation theory have established continuity properties of polynomial roots under bounded coefficient variations, ensuring preservation of root counts inside specified regions of the complex plane. Classical results on stability of polynomial roots under perturbations and analytic continuation arguments justify the extension of characteristic equation methods to interval coefficient settings. These theoretical developments collectively support the transformation of the fuzzy characteristic equation into α-level interval polynomials and provide the mathematical basis for root preservation and reconstruction of fuzzy stationary probabilities in finite buffer multi-server queues.

Recent advances in interval analysis and perturbation theory provide essential theoretical support for extending characteristic equation methods to uncertain parameter settings. The stability of polynomial roots under bounded coefficient perturbations has been extensively studied in numerical analysis and control theory, with results demonstrating continuity of root locations under compact coefficient variations [20, 5]. Such results are fundamental when characteristic equations contain interval-valued coefficients derived from α-cut decomposition. In parallel, interval polynomial analysis has been developed to determine root bounds and localization in complex domains, ensuring preservation of root counts within prescribed regions [9]. These theoretical insights justify the root-preservation arguments required for fuzzy $M^X/D/c/N$ systems.

Transformation-based approaches to waiting-time analysis also remain central in multi-server queueing theory. The Laplace–Stieltjes transform has long been used to derive waiting-time distributions in deterministic-service systems, and rational approximation techniques such as Padé approximants have been applied to approximate transform inversions when closed-form expressions are intractable [2]. Recent computational studies have emphasized transform-based numerical inversion methods for queueing models under structural complexity [1]. These transform-analytic techniques provide the methodological basis for extending waiting-time analysis into the fuzzy domain by performing α-level rational approximation and reconstructing interval-valued densities.

Recent studies also consider uncertainty modeling in conjunction with sustainable and stochastic decision making procedures in inventory and queuing systems. Behera [3] The work illustrates how the graded mean integration can convert fuzzy costing into practical decision steps in propagating epistemic uncertainty in sustainable supply chain systems. In a similar vein, Behera and Mohanta [4] presented a stochastic inventory model for crime resolution employing queueing theory and Markov decision processes with a twist by considering backlog as inventory and dynamically optimizing resolution policies with uncertainty. Their model combined Poisson arrivals, exponential service times, and monte carlo validation to model backlog behavior. These two developments, taken together, indicate an increasing emphasis on stochastic, fuzzy and dynamic optimization for complex service systems. However, neither study addressed the preservation of characteristic root structure in finite-buffer batch-arrival queues under fuzzy parameterization. The present work extends these methodological streams by incorporating α-cut interval decomposition into a roots-based $M^X/D/c/N$ analytical framework, thereby bridging fuzzy inventory theory, stochastic service systems, and classical generating-function queueing analysis.

From a purely theoretical perspective, fuzzy queueing models have now found applications in real service systems and a robust decision support mechanism to these models under parameter ambiguity was provided. In healthcare operations, fuzzy queueing models have been used to analyze patient wait times and capacity management in systems with uncertain estimates of service times and arrival rates [19]. In telecommunication and call center, a fuzzy two-server model has been introduced to obtain the staffing for a fuzzy arrival rate, which gives a range-valued number of servers instead of a single point value output [11]. Similarly, fuzzy finite-capacity queues have been applied to transportation and logistics systems to model demand uncertainties and service interruptions, allowing performance assessment in terms of uncertainty bands [6].

Moreover, optimization of queueing systems under fuzzy parameterization has been investigated using α-cut based nonlinear programming formulations and hybrid metaheuristic approaches, emphasizing the importance of computationally efficient uncertainty propagation mechanisms [13]. Collectively, these applied studies demonstrate that epistemic uncertainty substantially affects congestion risk, service reliability, and resource planning decisions. However, most existing applications rely on numerical approximation, recursive schemes, or simulation-based fuzzy evaluation, and do not preserve the intrinsic analytical structure of the underlying queueing model. In particular, the generating-function representation and characteristic-equation root structure, which enable exact stationary analysis in classical finite-buffer systems, are typically not retained in fuzzy extensions. Consequently, there remains considerable value in developing fuzzy queueing models that maintain structural analytic tractability while simultaneously delivering uncertainty-aware performance bands suitable for rigorous decision support.

To clearly position the novelty of the present work, Table 1 provides a structured comparison between representative existing approaches and the proposed fuzzy computational framework for the $M^X/D/c/N$ queue under parameter uncertainty.

Table 1. Comparative analysis of existing literature and the proposed mode.l

Study / Model Type	**Uncertainty Treatment**	**Analytical Structure**	**Limitations / Gap Addressed**
Classical $M^X/D/c/N$ models (e.g., roots-based methods)	Deterministic parameters	Characteristic equation and generating function methods	No treatment of epistemic uncertainty; assumes precise arrival and service parameters
Fuzzy $M/M/1$ and basic fuzzy queues	Triangular or L–R fuzzy numbers; α-cut decomposition	Recursive probability equations; numerical membership reconstruction	Mostly single-server; no preservation of characteristic root structure
Fuzzy inventory models with sustainability (e.g., Behera 2025)	Triangular fuzzy demand, holding cost, emission rates	Defuzzified cost optimization via graded mean method	Inventory context only; no queueing characteristic equation or waiting-time distribution analysis
Stochastic inventory / backlog models (e.g., Behera & Mohanta 2025)	Probabilistic uncertainty (Poisson arrivals, exponential service)	Markov Decision Process; Monte Carlo validation	No fuzzy epistemic modeling; no finite-buffer batch-arrival root analysis
Interval polynomial and perturbation theory	Bounded coefficient perturbation	Root stability and localization theory	Not applied to finite-buffer batch-arrival queueing systems
Proposed Fuzzy $M^X/D/c/N$ **Framework**	Triangular fuzzy arrival rate, service time, batch size	α-cut interval characteristic equation; root preservation; fuzzy generating function; fuzzy Laplace–Padé waiting-time analysis	Integrates fuzzy uncertainty with roots-based finite-buffer batch-arrival queue; preserves structural analytic properties while producing uncertainty bands for all performance measures

Despite substantial advances in both root-based analytic queueing theory and fuzzy queueing models, a significant methodological gap persists in unifying these two streams within a single rigorous framework. Classical characteristic-equation methods for finite-buffer batch-arrival systems possess a well-established structural elegance, enabling explicit derivation of stationary distributions and waiting-time characteristics through root localization inside the unit disk. These analytic ideas pair naturally with transform-based approximation and inversion tools such as Padé approximation and numerical Laplace inversion [2, 1]. However, such formulations assume exact knowledge of system parameters and do not account for epistemic uncertainty.

Conversely, the fuzzy queueing literature has developed numerous α-cut, interval, and optimization-based approaches for propagating parameter uncertainty into performance measures. The works by Chen [7], Panta et al. [17], Mueen [15], Liu et al. [12], and Rekha [18] are representative examples showing how interval decomposition and numerical reconstruction of membership functions can be used to handle triangular and L-R fuzzy numbers in queueing systems. However, these works are focused mostly on single-server systems or rely on recursive and heuristic methods that destroy the special root structure at the core of closed-form stationary analysis.

Parallel developments in interval analysis and perturbation theory, notably those associated with Kharitonov [9] and Rump [20], establish continuity and stability of polynomial roots under bounded coefficient variations. While these results provide the theoretical basis for root preservation under uncertainty, they have not been systematically integrated into the characteristic-equation methodology of finite-buffer batch-arrival queues.

Accordingly, there exists both theoretical and practical motivation to extend roots-based queueing analysis into a fuzzy environment by transforming the characteristic equation into α-level interval polynomials, establishing root-count preservation under a suitable fuzzy stability condition, computing interval enclosures of the characteristic roots across α levels, and reconstructing valid fuzzy stationary probabilities and performance bands. The present study addresses this gap by building directly upon the analytic structure of the crisp $M^X/D/c/N$ framework and integrating interval root-stability theory with α-cut arithmetic. The resulting formulation preserves structural tractability while providing uncertainty-aware performance measures, thereby yielding a mathematically rigorous and computationally implementable extension of finite-buffer bulk-arrival queueing theory.

2 | Preliminaries on Fuzzy Numbers

Queueing systems are not always uncertain because of randomness. Often, we lack exact probability data about things like arrival rates or service times because of missing info, expert guesses, or unclear conditions. In these cases, fuzziness can offer a solid mathematical way to show imprecision, without needing strict probabilities. So, in order to grasp the vague $M^X/D/c/N$ queue, first, we have to know what are fuzzy numbers and how we represent them. This part will cover the key definitions and properties we'll use later when we create the fuzzy calculation method.

Basic Definitions We begin with the formal definition of a fuzzy number.

Definition 1. *A fuzzy number $\tilde{A}$ on $\mathbb{R}$ is a fuzzy set characterized by a membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ satisfying the following properties:*

(i) ***Normality:*** *There exists $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$.*

(ii) ***Convexity:*** *For all $x, y \in \mathbb{R}$ and $\theta \in [0,1]$,*

$$\mu_{\tilde{A}}(\theta x + (1-\theta)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}.$$

(iii) ***Upper semi-continuity:*** *The membership function $\mu_{\tilde{A}}(x)$ is upper semi-continuous on $\mathbb{R}$.*

(iv) ***Compact support:*** *The closure of the set $\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\}$ is compact.*

The previous definition ensures that any fuzzy number has bounded level sets and can be seen as an interval valued generalization of a real number. Convexity of the membership ensures that middle values do not have a smaller membership than extreme values.It is, on the other hand, very important to have interval-based computations via α-level sets.

Among various classes of fuzzy numbers, triangular fuzzy numbers are particularly useful due to their analytical simplicity and computational tractability.

Definition 2. *A triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$ is defined by the membership function*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3, \\ 0, & \textit{otherwise}. \end{cases}$$

Here a_2 represents the most plausible value, while a_1 and a_3 denote the lower and upper bounds of possible variation respectively. The triangular structure allows closed form expressions for α level sets and significantly simplifies the computational algorithm developed later for the fuzzy characteristic equation.

α Cut Representation The α-cut method is key to changing fuzzy problems into a set of interval-based problems. Because the characteristic equation of the model and the steady state probabilities are defined using algebra, changing fuzzy parameters into interval form at each confidence level $\alpha \in [0,1]$ is simple from an analysis point of view. This lets one study the fuzzy $M^X/D/c/N$ queue by using deterministic interval analysis and then rebuild it as a fuzzy solution.

Definition 3. *Let $\tilde{A}$ be a fuzzy number with membership function $\mu_{\tilde{A}}(x)$. For $\alpha \in [0,1]$, the α cut of $\tilde{A}$ is defined as*

$$A^\alpha = \{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\}.$$

For a fuzzy number satisfying the properties stated earlier, each α cut is a closed bounded interval which may be written as

$$A^\alpha = [A_L^\alpha, A_U^\alpha],$$

where A_L^α and A_U^α denote the lower and upper bounds respectively.

Triangular Case. If $\tilde{A} = (a_1, a_2, a_3)$ is a triangular fuzzy number, then its α cut admits the explicit representation

$$A^\alpha = [a_1 + \alpha(a_2 - a_1),\, a_3 - \alpha(a_3 - a_2)], \qquad \alpha \in [0,1].$$

Hence at $\alpha = 0$ we get the support interval $[a_1, a_3]$, and at $\alpha = 1$ we get the singleton $\{a_2\}$ of the most plausible value.

Arithmetic via α-Cuts.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers with α cuts

$$A^\alpha = [A_L^\alpha, A_U^\alpha], \qquad B^\alpha = [B_L^\alpha, B_U^\alpha].$$

The elementary arithmetic operations are levelled and defined by:

$$(\tilde{A} + \tilde{B})^\alpha = [A_L^\alpha + B_L^\alpha,\ A_U^\alpha + B_U^\alpha]\,,$$

$$(\tilde{A} - \tilde{B})^\alpha = [A_L^\alpha - B_U^\alpha,\ A_U^\alpha - B_L^\alpha]\,,$$

$$(\tilde{A} \times \tilde{B})^\alpha = [\min \mathsf{S}^\alpha,\ \max \mathsf{S}^\alpha]\,,$$

where

$$\mathsf{S}^\alpha = \{A_L^\alpha B_L^\alpha, A_L^\alpha B_U^\alpha, A_U^\alpha B_L^\alpha, A_U^\alpha B_U^\alpha\}\,.$$

When the fuzzy numbers are strictly positive, the multiplication reduces to

$$(\tilde{A} \times \tilde{B})^\alpha = [A_L^\alpha B_L^\alpha,\ A_U^\alpha B_U^\alpha]\,.$$

Division.

If $0 \notin B^\alpha$ for all $\alpha \in [0,1]$, then

$$\left(\frac{\tilde{A}}{\tilde{B}}\right)^\alpha = [\min \mathsf{T}^\alpha,\ \max \mathsf{T}^\alpha]\,,$$

where

$$\mathsf{T}^\alpha = \left\{\frac{A_L^\alpha}{B_L^\alpha}, \frac{A_L^\alpha}{B_U^\alpha}, \frac{A_U^\alpha}{B_L^\alpha}, \frac{A_U^\alpha}{B_U^\alpha}\right\}.$$

These interval operations ensure that all algebraic manipulations required for the fuzzy characteristic equation and the fuzzy traffic intensity can be carried out through deterministic interval computations at each α level. The fuzzy solution is then reconstructed by assembling the family of interval solutions over $\alpha \in [0,1]$.

Nesting Property and Reconstruction of Fuzzy Numbers The validity of the α cut approach relies fundamentally on the nesting structure of level sets. When fuzzy parameters are propagated through algebraic and functional operations, the resulting interval solutions at each α level must preserve this nesting property in order to represent a well defined fuzzy number. This subsection formalizes the necessary structural conditions that ensure the correctness of the reconstruction procedure.

Definition 4. *Let $\tilde{A}$ be a fuzzy number with α cuts $A^\alpha = [A_L^\alpha, A_U^\alpha]$. The family $\{A^\alpha : \alpha \in [0,1]\}$ is said to satisfy the nesting property if for all $0 \le \alpha_1 \le \alpha_2 \le 1$,*

$$A^{\alpha_2} \subseteq A^{\alpha_1}.$$

The nesting property ensures that at higher confidence levels one gets smaller intervals, in agreement with the interpretation of α as a level of membership.

For Triangular fuzzy numbers nesting property is a direct consequence of monotonicity of the endpoint functions.

Proposition 1. *Let $\tilde{A} = (a_1, a_2, a_3)$ be a triangular fuzzy number (TFN). Then the lower endpoint A_L^α is nondecreasing in α, and the upper endpoint A_U^α is nonincreasing in α. Consequently, the α cut family satisfies the nesting property.*

Proof. From the explicit representation,

$$A_L^\alpha = a_1 + \alpha(a_2 - a_1), \qquad A_U^\alpha = a_3 - \alpha(a_3 - a_2).$$

As $a_2 \geq a_1$, the function A_L^α is linear and monotone increasing in α. By the same token, $a_3 \geq a_2$ implies that A_U^α is linear and monotone decreasing in α. Therefore, for $\alpha_1 \leq \alpha_2$,

$$A_L^{\alpha_1} \leq A_L^{\alpha_2} \quad \text{and} \quad A_U^{\alpha_1} \geq A_U^{\alpha_2},$$

which implies

$$[A_L^{\alpha_2}, A_U^{\alpha_2}] \subseteq [A_L^{\alpha_1}, A_U^{\alpha_1}].$$

□

The following result ensures that the reconstruction of fuzzy stationary probabilities from interval solutions is mathematically valid.

Theorem 1. *Suppose that for each $\alpha \in [0,1]$, a closed bounded interval $P^\alpha = [P_L^\alpha, P_U^\alpha]$ is obtained such that*

(1) P_L^α is nondecreasing in α,
(2) P_U^α is nonincreasing in α,
(3) $P_L^\alpha \leq P_U^\alpha$ for all α,
(4) P^1 is a singleton.

Then there exists a unique fuzzy number $\tilde{P}$ whose α cuts coincide with P^α.

Proof. The stated conditions ensure that the family $\{P^\alpha\}$ forms a nested, closed, and bounded collection of intervals satisfying the standard representation theorem for fuzzy numbers. Using the resolution identity, one may define the membership function

$$\mu_{\tilde{P}}(x) = \sup\{\alpha \in [0,1] : x \in P^\alpha\}.$$

The nesting conditions guarantee that this membership function is well defined, convex, upper semicontinuous, and normalized. Therefore $\tilde{P}$ is a fuzzy number with the prescribed α cuts. □

The above theorem provides the theoretical foundation for the computational strategy adopted later. In particular, when solving the fuzzy characteristic equation and deriving stationary probabilities, we shall compute interval solutions at discretized α levels. As long as the resulting interval endpoints satisfy the monotonicity conditions established above, the reconstructed fuzzy stationary probabilities and performance measures will constitute valid fuzzy numbers.

Fuzzy Functions and Interval Polynomial Roots The stationary distribution of the classical $M^X/D/c/N$ queue is obtained through the solution of a characteristic equation involving analytic functions of system parameters. When these parameters become fuzzy, the characteristic equation itself becomes a fuzzy functional equation. In order to preserve mathematical tractability, we transform this fuzzy functional problem into a family of interval polynomial equations through the α cut decomposition. This subsection formalizes the necessary concepts.

Fuzzy Valued Functions.

Definition 5. *Let $\tilde{\theta}$ be a fuzzy number with α cuts $\theta^\alpha = [\theta_L^\alpha, \theta_U^\alpha]$. A fuzzy valued function $\tilde{F}(x, \tilde{\theta})$ is defined through its α cut representation as*

$$F^\alpha(x) = \{F(x, \theta) : \theta \in [\theta_L^\alpha, \theta_U^\alpha]\}.$$

Therefore, for each fixed α, the fuzzyfunction is an interval valued function. If $F(x, \theta)$ is continuous in θ, then $F^\alpha(x)$ is a closed interval given by

$$F^\alpha(x) = \min_{\theta \in \theta^\alpha} F(x, \theta), \ \max_{\theta \in \theta^\alpha} F(x, \theta) \ .$$

Interval Polynomial Representation.

In the fuzzy $M^X/D/c/N$ model, the characteristic equation has the general form

$$z^c \quad \tilde{\pi}^*(z, \tilde{D}) = 0,$$

where $\tilde{D}$ denotes the fuzzy service time and $\tilde{\pi}^*(z, \tilde{D})$ depends continuously on $\tilde{D}$.

For each $\alpha \in [0, 1]$, the equation reduces to an interval polynomial equation

$$z^c \quad \pi_\alpha^*(z) = 0,$$

where

$$\pi_\alpha^*(z) = \min_{D \in D^\alpha} \pi^*(z, D), \ \max_{D \in D^\alpha} \pi^*(z, D) \ .$$

Thus, the fuzzy characteristic equation is equivalent to solving a family of deterministic polynomial equations with interval coefficients.

Interval Roots.

Definition 6. *Let*

$$P^\alpha(z) = z^c \quad \pi_\alpha^*(z)$$

be an interval polynomial at level α. A complex number z is said to be an admissible α level root if there exists a coefficient realization within the interval bounds such that $P^\alpha(z) = 0$.

This notion corresponds to the classical concept of roots of interval polynomials. In practical computation, one determines the set of roots associated with the lower and upper bound polynomials and then constructs interval enclosures of roots.

Root Continuity Under Parameter Perturbation.

The following stability result ensures that small uncertainty in parameters does not lead to discontinuous jumps in the root structure.

Theorem 2. *Suppose $P(z, \theta)$ is analytic in z and continuous in θ. If for a crisp parameter θ_0 the equation $P(z, \theta_0) = 0$ has exactly c roots inside the unit disk and none on the boundary, then there exists $\varepsilon > 0$ such that for all θ satisfying $|\theta - \theta_0| < \varepsilon$, the number of roots inside the unit disk remains c.*

Proof. The result follows from Rouché's theorem and continuity of analytic functions under coefficient perturbations. Since roots of analytic polynomials depend continuously on coefficients, sufficiently small perturbations preserve the root count within any closed contour not intersecting roots. ☐

This theorem provides the theoretical justification for extending the classical root-based method to the fuzzy setting. At each α level, provided the interval bounds remain within the stability region, the number of roots inside the unit disk remains invariant. Consequently, the fuzzy stationary distribution can be computed by solving interval root problems at discretized α levels and reconstructing the fuzzy probabilities through the nesting property established earlier.

3 | Model Formulation

This section develops the fuzzy $M^X/D/c/N$ queue under parameter uncertainty. We begin by recalling the structure of the classical model in order to fix notation and highlight the quantities that will later be generalized to fuzzy counterparts. Subsequently, uncertain parameters are modeled as triangular fuzzy numbers and the corresponding fuzzy traffic intensity is defined. A fundamental stability theorem is then established, providing the necessary and sufficient condition for the existence of a fuzzy steady state distribution.

Classical Model Consider a finite buffer multi-server queue in which customers arrive in batches according to a Poisson process with rate $\lambda > 0$. Let X denote the batch size with probability mass function

$$P(X = k) = g_k, \qquad k = 1, 2, \ldots,$$

and mean batch size

$$g = \sum_{k=1}^{\infty} k g_k.$$

The service time of each customer is deterministic and equal to $D > 0$. There are c identical parallel servers, and the waiting room capacity is N, excluding customers in service.

The offered load or traffic intensity of the system is given by

$$\rho = \frac{\lambda g D}{c}.$$

It is well known that the classical $M^X/D/c/N$ queue admits a steady state distribution when $\rho < 1$. In this case, the system length probabilities can be derived using the roots of the characteristic equation associated with the probability generating function.

Fuzzy Parameter Specification In practical systems, the parameters λ, D, and g may not be known precisely. Alternatively, they can be approximated by expert opinion or partial observations. To reflect this uncertainty, we consider these parameters as triangular fuzzy numbers.

Let

$$\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3), \qquad \tilde{D} = (D_1, D_2, D_3), \qquad \tilde{g} = (g_1, g_2, g_3),$$

be triangular fuzzy numbers representing the uncertain arrival rate, service time, and mean batch size respectively.

For each α 2 $[0, 1]$, their α cuts are given by

$$\lambda^\alpha = [\lambda_1 + \alpha(\lambda_2 \quad \lambda_{\ 1}),\ \lambda_3 \quad \alpha(\lambda_{\ 3} \quad \lambda_{\ 2})]\,,$$

$$D^\alpha = [D_1 + \alpha(D_2 \quad D_{\ 1}),\ D_3 \quad \alpha(D_{\ 3} \quad D_{\ 2})]\,,$$

$$g^\alpha = [g_1 + \alpha(g_2 \quad g_{\ 1}),\ g_3 \quad \alpha(g_{\ 3} \quad g_{\ 2})]\,.$$

Fuzzy Traffic Intensity.

The fuzzy traffic intensity is defined as

$$\tilde{\rho} = \frac{\tilde{\lambda}\tilde{g}\tilde{D}}{c}.$$

Since $\tilde{\lambda}$, $\tilde{g}$, and $\tilde{D}$ are assumed to be positive fuzzy numbers, the α cut of $\tilde{\rho}$ is given by

$$\rho^\alpha = \quad \frac{\lambda_L^\alpha g_L^\alpha D_L^\alpha}{c}, \frac{\lambda_U^\alpha g_U^\alpha D_U^\alpha}{c} \quad .$$

Thus, for each α, the fuzzy system reduces to a deterministic interval valued traffic intensity.

The following theorem establishes the fundamental stability condition for the fuzzy queue.

Theorem 3. *If for every α 2 $[0, 1]$ the upper bound of the α cut of the traffic intensity satisfies*

$$\rho_U^\alpha < 1,$$

then the fuzzy $M^X/D/c/N$ queue admits a steady state distribution in the sense that the stationary probabilities at each α level exist and form nested intervals.

Proof. For each fixed α 2 $[0, 1]$, the fuzzy parameters reduce to deterministic intervals λ^α, D^α, and g^α. Consider any realization (λ, D, g) within these intervals. The corresponding crisp traffic intensity satisfies

$$\rho = \frac{\lambda g D}{c} \quad \rho^{\ \alpha}_{\ U}.$$

By assumption, $\rho_U^\alpha < 1$, hence for every admissible realization within the α level intervals, the classical stability condition $\rho < 1$ holds. Therefore, for each α, the associated crisp $M^X/D/c/N$ queue admits a unique steady state distribution.

Since the stability condition holds uniformly over all realizations within the α level intervals, the set of stationary probabilities at level α forms a closed bounded interval. Furthermore, because the α cuts of the parameters are nested, the resulting stationary probability intervals inherit the nesting property established in the previous section.

Hence, by the reconstruction theorem for fuzzy numbers, the family of interval stationary probabilities defines a valid fuzzy steady state distribution. □

Fuzzy Characteristic Equation The stationary distribution of the classical $M^X/D/c/N$ queue is obtained through the probability generating function method. In particular, the unknown stationary probabilities are determined by the roots of a characteristic equation of the form

$$z^c - \pi^*(z, D) = 0,$$

where $\pi^*(z, D)$ denotes the probability generating function of arrivals during a service time of length D.

When the service time and arrival rate become fuzzy, the characteristic equation itself becomes fuzzy. In this subsection, we construct the fuzzy characteristic equation and reduce it to a family of interval polynomial equations using the α cut representation.

Fuzzy Arrival Generating Function.

For the classical model, the generating function of arrivals during a deterministic service time D is given by

$$\pi^*(z, D) = \exp\bigl(\lambda D(G(z) - 1)\bigr),$$

where

$$G(z) = \sum_{k=1}^{\infty} g_k z^k$$

is the probability generating function of the batch size distribution.

Under parameter uncertainty, λ, D, and g become fuzzy numbers. Hence we define the fuzzy generating function

$$\tilde{\pi}^*(z) = \exp\bigl(\tilde{\lambda}\tilde{D}(\tilde{G}(z) - 1)\bigr).$$

Since the exponential function is continuous and monotone in its argument, its α cut can be obtained through interval evaluation.

α Level Representation.

For each $\alpha \in [0, 1]$, let

$$\lambda^\alpha = [\lambda_L^\alpha, \lambda_U^\alpha], \quad D^\alpha = [D_L^\alpha, D_U^\alpha], \quad g^\alpha = [g_L^\alpha, g_U^\alpha].$$

Then the interval valued generating function at level α is given by

$$\pi_\alpha^*(z) = \left[\min_{\lambda,D,g} \exp\bigl(\lambda D(G(z) - 1)\bigr),\ \max_{\lambda,D,g} \exp\bigl(\lambda D(G(z) - 1)\bigr)\right],$$

where the minimization and maximization are taken over the Cartesian product of the α level intervals.

Since all parameters are positive and $(G(z) - 1)$ is fixed for a given z, monotonicity implies

$$\pi_\alpha^*(z) = \bigl[\exp\bigl(\lambda_L^\alpha D_L^\alpha (G_L^\alpha(z) - 1)\bigr),\ \exp\bigl(\lambda_U^\alpha D_U^\alpha (G_U^\alpha(z) - 1)\bigr)\bigr].$$

Interval Characteristic Equation.

The fuzzy characteristic equation is defined as

$$z^c - \tilde{\pi}^*(z) = 0.$$

At each α level, this reduces to the interval equation

$$z^c - \pi^*_\alpha(z) = 0,$$

which may be interpreted as the family of deterministic equations

$$z^c - \pi^*(z, \theta) = 0, \quad \theta \in \Theta^\alpha,$$

where Θ^α denotes the admissible parameter region at level α.

Root Structure Preservation.

The following theorem guarantees that the roots based computational procedure remains valid under fuzziness.

Theorem 4. *Suppose that for every $\alpha \in [0, 1]$ the upper bound of the traffic intensity satisfies $\rho_U^\alpha < 1$. Then for each α, the interval characteristic equation*

$$z^c - \pi^*_\alpha(z) = 0$$

admits exactly c roots inside the unit disk and no root on the boundary.

Proof. For each fixed α, every admissible parameter realization satisfies the classical stability condition $\rho < 1$. From the known root localization result for the crisp $M^X/D/c$ model, the characteristic equation admits exactly c roots inside the unit disk. Since the coefficients depend continuously on parameters and the admissible parameter region at level α is compact, the root count inside the unit disk remains invariant under all admissible perturbations. Hence the interval characteristic equation admits precisely c admissible roots inside the unit disk.

This result establishes that the fuzzy stationary distribution may be computed by solving the characteristic equation at discretized α levels and determining the corresponding interval enclosures of roots. The stationary probabilities are then obtained using the same algebraic structure as in the classical roots based method, followed by reconstruction through the nesting property.

Fuzzy Stationary Probability Generating Function Having established that the fuzzy characteristic equation preserves the root structure at each α level, we now derive the fuzzy analogue of the stationary probability generating function. The development closely follows the classical roots based paradigm, with the essential difference that all quantities are interpreted through their α level interval representations.

Classical Generating Function Structure.

For the crisp $M^X/D/c/N$ queue under $\rho < 1$, the stationary system length probabilities $\{p_i\}$ satisfy the probability generating function identity

$$P^*(z) = \frac{\pi^*(z) \sum_{i=0}^{c-1} p_i (z^c - z^i)}{z^c - \pi^*(z)}.$$

The unknown probabilities $p_0, p_1, \ldots, p_{c-1}$ are determined by matching the zeros of the numerator and denominator corresponding to the c roots inside the unit disk, together with the normalization condition.

Fuzzy Generating Function.

Under parameter uncertainty, we define the fuzzy stationary probability generating function

$$\tilde{P}^*(z) = \frac{\tilde{\pi}^*(z) \sum_{i=0}^{c-1} \tilde{p}_i (z^c - z^i)}{z^c - \tilde{\pi}^*(z)}.$$

For each $\alpha \in [0,1]$, this reduces to the interval valued representation

$$P_\alpha^*(z) = \frac{\pi_\alpha^*(z) \sum_{i=0}^{c-1} p_i^\alpha (z^c - z^i)}{z^c - \pi_\alpha^*(z)}.$$

Thus, at every α level, the problem becomes that of determining interval stationary probabilities p_i^α.

Determination of Fuzzy Stationary Probabilities.

Let $\eta_1^\alpha, \dots, \eta_c^\alpha$ denote the c roots inside the unit disk of the interval characteristic equation at level α. Then, for each α, the unknown probabilities satisfy the system

$$\sum_{i=0}^{c-1} p_i^\alpha \left((\eta_j^\alpha)^c - (\eta_j^\alpha)^i \right) = 0, \qquad j = 2, \dots, c,$$

together with the normalization condition

$$\sum_{i=0}^{c+N} p_i^\alpha = 1.$$

Since the coefficient matrices depend continuously on parameters and the admissible parameter region at level α is compact, the solution set of this linear system forms a bounded interval for each p_i.

Theorem 5. *Suppose that for every $\alpha \in [0,1]$ the stability condition $\rho_U^\alpha < 1$ holds. Then for each α, the stationary probabilities p_i^α exist uniquely and form closed bounded intervals. Moreover, the family $\{p_i^\alpha : \alpha \in [0,1]\}$ satisfies the nesting property, and hence defines a fuzzy stationary probability $\tilde{p}_i$.*

Proof. For each fixed α, the admissible parameter region is compact and satisfies the classical stability condition. Therefore, the crisp stationary probabilities exist uniquely for every admissible parameter realization.

Since the generating function and the associated linear system depend continuously on parameters, the stationary probabilities vary continuously with parameters. Hence the image of the compact parameter region under the stationary probability mapping is a compact interval. This establishes existence and boundedness of p_i^α.

Because the α cuts of the parameters are nested, the corresponding parameter regions shrink as α increases. Therefore, the induced stationary probability intervals also shrink monotonically, implying the nesting property.

By the reconstruction theorem established earlier, the family $\{p_i^\alpha\}$ defines a unique fuzzy number $\tilde{p}_i$. □

Fuzzy Queue Length Distribution.

Let $\tilde{q}_k$ denote the fuzzy probability that k customers are present in the queue excluding those in service. Then

$$\tilde{q}_0 = \sum_{i=0}^{c} \tilde{p}_i, \qquad \tilde{q}_k = \tilde{p}_{c+k}, \quad 1 \le k \le N.$$

All the performance measures in term of queue length e.g. fuzzy average queue length and fuzzy blocking probability can be obtained immediately from these fuzzy stationary probabilities.

This concludes the derivation of the fuzzy steady state distribution via the roots based computational method. The subsequent step is to generalize the classical performance measures such as mean queue length and waiting time distribution to the fuzzy setting.

4 | Fuzzy Characteristic Equation

The roots based computational paradigm for the classical $M^X/D/c/N$ queue relies fundamentally on the characteristic equation associated with the probability generating function of the system length distribution. In the fuzzy setting, the uncertainty in parameters induces uncertainty in this characteristic equation. The objective of this section is to construct the fuzzy characteristic equation and establish that the essential root structure required for the stationary distribution is preserved under parameter uncertainty.

Construction For the classical model with deterministic service time D, the generating function of arrivals during one service period is given by

$$\pi^*(z, D) = \exp \; \lambda D(G(z) - 1) \; ,$$

where

$$G(z) = \sum_{k=1}^{\infty} g_k z^k$$

denotes the batch size probability generating function.

The characteristic equation that determines the stationary probabilities is

$$z^c - \pi^*(z, D) = 0.$$

Under parameter uncertainty, λ, D, and g are modeled as fuzzy numbers. Consequently, the generating function becomes fuzzy and is denoted by

$$\tilde{\pi}^*(z, \tilde{D}) = \exp \; \tilde{\lambda}\tilde{D}(\tilde{G}(z) - 1) \; .$$

The fuzzy characteristic equation is therefore defined as

$$z^c - \tilde{\pi}^*(z, \tilde{D}) = 0.$$

Since $\tilde{\pi}^*(z, \tilde{D})$ is a fuzzy valued function, this equation must be interpreted via its α cut representation.

α Level Decomposition For each $\alpha \in [0, 1]$, the fuzzy parameters admit interval representations

$$\lambda^\alpha = [\lambda_L^\alpha, \lambda_U^\alpha], \quad D^\alpha = [D_L^\alpha, D_U^\alpha], \quad g^\alpha = [g_L^\alpha, g_U^\alpha].$$

The corresponding interval generating function at level α is

$$\pi_\alpha^*(z) = \; \min_{\theta \in \Theta^\alpha} \pi^*(z, \theta), \; \max_{\theta \in \Theta^\alpha} \pi^*(z, \theta) \; ,$$

where Θ^α denotes the admissible parameter region.

Thus, the fuzzy characteristic equation reduces at each α level to the interval polynomial equation

$$z^c - \pi^*_\alpha(z) = 0.$$

Equivalently, one considers the family of deterministic equations

$$z^c - \pi^*(z, \theta) = 0, \qquad \theta \in \Theta^\alpha.$$

The following theorem guarantees preservation of the fundamental root structure under fuzziness.

Theorem 6. *Suppose that for every $\alpha \in [0, 1]$ the upper bound of the traffic intensity satisfies*

$$\rho^\alpha_U < 1.$$

Then, for each α, the interval polynomial equation

$$z^c - \pi^*_\alpha(z) = 0$$

admits exactly c roots inside the unit disk and no root on the boundary.

Proof. Fix $\alpha \in [0, 1]$. By the assumed fuzzy stability condition, every admissible parameter realization at level α satisfies the classical stability condition $\rho < 1$. For the crisp $M^X/D/c$ model under $\rho < 1$, it is known that the characteristic equation

$$z^c - \pi^*(z, D) = 0$$

admits exactly c roots inside the unit disk and none on the boundary.

Since $\pi^*(z, \theta)$ depends continuously on parameters and the admissible parameter region Θ^α is compact, the coefficients of the polynomial vary continuously within bounded intervals. By continuity of roots with respect to polynomial coefficients and classical root localization results such as Rouché's theorem, the number of roots inside the unit disk remains invariant under such bounded perturbations.

Therefore, the interval polynomial equation at level α admits exactly c admissible roots inside the unit disk and none on the boundary. □

This result establishes that the roots based computational method extends naturally to the fuzzy setting. At each α level, the stationary distribution may be computed by solving the associated interval characteristic equation and then reconstructing the fuzzy stationary probabilities through the nesting property.

5 | Fuzzy Stationary Distribution

Having established that the fuzzy characteristic equation preserves the required root structure at each α level, we now construct the fuzzy stationary distribution. The derivation follows the classical generating function approach, but every step is interpreted through α cut analysis. The goal is to demonstrate that the stationary probabilities obtained under parameter uncertainty form well defined fuzzy numbers satisfying normalization and nesting properties.

Fuzzy Probability Generating Function For the classical $M^X/D/c/N$ queue, the stationary system length probabilities satisfy the generating function identity

$$P^*(z) = \frac{\pi^*(z,D)\sum_{i=0}^{c-1} p_i(z^c - z^i)}{z^c - \pi^*(z,D)}.$$

Under fuzziness, the corresponding generating function is defined as

$$\tilde{P}^*(z) = \frac{\tilde{\pi}^*(z,\tilde{D})\sum_{i=0}^{c-1} \tilde{p}_i(z^c - z^i)}{z^c - \tilde{\pi}^*(z,\tilde{D})}.$$

For each $\alpha \in [0,1]$, this reduces to the interval representation

$$P^*_\alpha(z) = \frac{\pi^*_\alpha(z)\sum_{i=0}^{c-1} p_i^\alpha(z^c - z^i)}{z^c - \pi^*_\alpha(z)}.$$

Let $\eta_1^\alpha, \dots, \eta_c^\alpha$ denote the c roots inside the unit disk of the interval characteristic equation at level α. The unknown probabilities $p_0^\alpha, \dots, p_{c-1}^\alpha$ are determined by solving the system

$$\sum_{i=0}^{c-1} p_i^\alpha \left((\eta_j^\alpha)^c - (\eta_j^\alpha)^i \right) = 0, \qquad j = 2, \dots, c,$$

together with the normalization condition

$$\sum_{i=0}^{c+N} p_i^\alpha = 1.$$

Thus, at each α level, the stationary probabilities are obtained by solving a deterministic linear system whose coefficients depend continuously on the interval parameters.

Existence and Uniqueness We now establish that the fuzzy stationary probabilities obtained via α level computation form valid fuzzy numbers.

Theorem 7. *Suppose that for every $\alpha \in [0,1]$ the fuzzy stability condition*

$$\rho_U^\alpha < 1$$

holds. Then the stationary probabilities $\tilde{p}_i$ defined through their α cuts

$$p_i^\alpha = [p_{i,L}^\alpha, p_{i,U}^\alpha]$$

exist uniquely for each i and constitute valid fuzzy numbers.

Proof. Fix $\alpha \in [0,1]$. Under the stability condition $\rho_U^\alpha < 1$, every admissible parameter realization satisfies the classical condition for existence and uniqueness of the stationary distribution. Hence, for each admissible realization, the stationary probabilities are uniquely determined.

Since the generating function and the associated linear system depend continuously on parameters, the mapping from parameter space to stationary probabilities is continuous. Because the admissible parameter region at level α is compact, the image of this mapping is also compact. Therefore, for each i, the set of admissible stationary probabilities at level α forms a closed bounded interval.

As α increases, the admissible parameter region shrinks due to the nesting property of α cuts. Consequently, the corresponding stationary probability intervals also shrink monotonically. Thus, the family $\{p_i^\alpha : \alpha \in [0,1]\}$ satisfies the nesting conditions required for reconstruction of a fuzzy number.

Hence, each $\tilde{p}_i$ defined through its α cuts is a valid fuzzy number. $\square$

6 | Fuzzy Performance Measures

Once the fuzzy stationary probabilities have been established, all classical performance measures can be extended to the fuzzy setting through α level arithmetic. Because the stationary probabilities $\tilde{p}_i$ and queue probabilities $\tilde{q}_i$ are valid fuzzy numbers, any performance measure defined as a continuous functional of these quantities admits a well defined fuzzy representation. In this section, we derive the fuzzy mean queue length and fuzzy mean waiting time, and establish their existence under the fuzzy stability condition.

Fuzzy Mean Queue Length.

Let $\tilde{q}_k$ denote the fuzzy probability that k customers are present in the queue excluding those in service. The fuzzy mean queue length is defined as

$$\tilde{L}_q = \sum_{k=0}^{N} k\tilde{q}_k.$$

For each $\alpha \in [0,1]$, the α cut of $\tilde{L}_q$ is given by

$$L_q^\alpha = \left[\sum_{k=0}^{N} kq_{k,L}^\alpha,\ \sum_{k=0}^{N} kq_{k,U}^\alpha\right],$$

since all coefficients k are nonnegative and the α level queue probabilities are nonnegative intervals.

Proposition 2. *If the fuzzy stability condition $\rho_U^\alpha < 1$ holds for all $\alpha \in [0,1]$, then $\tilde{L}_q$ exists uniquely and is a valid fuzzy number.*

Proof. For each α, the stationary probabilities q_k^α form bounded intervals satisfying normalization and nonnegativity. The function

$$(q_0, \ldots, q_N) \mapsto \sum_{k=0}^{N} kq_k$$

is linear and continuous. Therefore, the image of the compact interval vector $(q_0^\alpha, \ldots, q_N^\alpha)$ under this mapping is a compact interval. Since the α cuts of q_k are nested, the resulting intervals L_q^α are also nested. Hence $\tilde{L}_q$ is a well defined fuzzy number. $\square$

Effective Fuzzy Arrival Rate.

Because the system has finite capacity, some arrivals may be blocked. Let $\tilde{P}_{\text{loss}}$ denote the fuzzy blocking probability. The fuzzy effective arrival rate is defined as

$$\tilde{\lambda}_{\text{eff}} = \tilde{\lambda}\left(1 - \tilde{P}_{\text{loss}}\right).$$

For each α level,

$$\lambda_{\text{eff}}^\alpha = \left[\lambda_L^\alpha(1 - P_{\text{loss},U}^\alpha),\ \lambda_U^\alpha(1 - P_{\text{loss},L}^\alpha)\right],$$

since both $\tilde{\lambda}$ and $(1 - \tilde{P}_{\text{loss}})$ are positive fuzzy numbers.

Fuzzy Mean Waiting Time.

By Little’s law, the classical mean waiting time in queue is

$$W_q = \frac{L_q}{\lambda_{\text{eff}}}.$$

The fuzzy mean waiting time is therefore defined as

$$\tilde{W}_q = \frac{\tilde{L}_q}{\tilde{\lambda}_{\text{eff}}}.$$

At each $\alpha \in [0,1]$, the α cut of $\tilde{W}_q$ is obtained through interval division:

$$W_q^{\alpha} = \left[\min \mathcal{S}^{\alpha},\ \max \mathcal{S}^{\alpha}\right],$$

where

$$\mathcal{S}^{\alpha} = \left\{ \frac{L^{\alpha}_{q,L}}{\lambda^{\alpha}_{\text{eff},L}}, \frac{L^{\alpha}_{q,L}}{\lambda^{\alpha}_{\text{eff},U}}, \frac{L^{\alpha}_{q,U}}{\lambda^{\alpha}_{\text{eff},L}}, \frac{L^{\alpha}_{q,U}}{\lambda^{\alpha}_{\text{eff},U}} \right\}.$$

Theorem 8. *Under the fuzzy stability condition and assuming $\lambda^{\alpha}_{\text{eff},L} > 0$ for all $\alpha \in [0,1]$, the fuzzy mean waiting time $\tilde{W}_q$ exists uniquely and is a valid fuzzy number.*

Proof. For each α, the intervals L_q^{α} and $\lambda^{\alpha}_{\text{eff}}$ are compact and satisfy positivity of the denominator. The function

$$(L_q, \lambda_{\text{eff}}) \mapsto \frac{L_q}{\lambda_{\text{eff}}}$$

is continuous on the positive domain. Hence its image over the compact interval product is a compact interval. Since the α level intervals shrink monotonically due to nesting of parameters, the resulting waiting time intervals also shrink monotonically. Therefore, the family $\{W_q^{\alpha}\}$ satisfies the nesting property and defines a valid fuzzy number.

Thus, all primary performance measures of the $M^X/D/c/N$ queue admit consistent fuzzy generalizations under the proposed computational framework.

7 | Fuzzy Waiting Time Distribution

The stationary mean waiting time derived in the previous section provides only first moment information. In many applications, however, the full waiting time distribution is required for risk assessment and service level guarantees. In the classical $M^X/D/c/N$ model, the waiting time distribution can be obtained through the distributional form of Little’s law and Laplace transform techniques. In the presence of parameter uncertainty, these transforms become fuzzy valued. This section develops the fuzzy waiting time distribution using α level Laplace transform analysis and rational approximation.

Fuzzy Laplace Transform Let $W_q(t)$ denote the waiting time in queue of a randomly selected customer. In the crisp model, its Laplace transform is defined as

$$W_q^{*}(s) = \int_0^{\infty} e^{-st}\, dW_q(t).$$

Under parameter uncertainty, the waiting time becomes fuzzy and is denoted by $\tilde{W}_q(t)$. We define the fuzzy Laplace transform as

$$\tilde{W}_q^{*}(s) = \int_0^{\infty} e^{-st}\, d\tilde{W}_q(t).$$

Using the distributional form of Little's law, the classical relation between the queue length generating function and the waiting time Laplace transform yields

$$\tilde{W}_q^*(s) = \tilde{P}(s),$$

where $\tilde{P}(s)$ denotes the corresponding fuzzy transform expression derived from the fuzzy stationary probabilities.

For each $\alpha \in [0,1]$, the α cut of the Laplace transform is given by

$$W_{q,\alpha}^*(s) = \left[\min_{\theta \in \Theta^\alpha} W_q^*(s,\theta),\ \max_{\theta \in \Theta^\alpha} W_q^*(s,\theta)\right],$$

where Θ^α denotes the admissible parameter region at level α.

Since the Laplace transform depends continuously on parameters and the parameter region is compact, $W_{q,\alpha}^*(s)$ is a closed bounded interval for each $s \geq 0$.

Fuzzy Padé Approximation In the classical model, the Laplace transform $W_q^*(s)$ is often expressed as a rational function obtained via Padé approximation,

$$W_q^*(s) \approx \frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials.

Under fuzziness, for each α level we approximate

$$W_{q,\alpha}^*(s) \approx \frac{P_\alpha(s)}{Q_\alpha(s)},$$

where $P_\alpha(s)$ and $Q_\alpha(s)$ are interval coefficient polynomials determined from the α level data.

Because the Padé approximation is constructed from a finite number of moments and these moments vary continuously with parameters, the resulting polynomial coefficients form bounded intervals.

The inverse Laplace transform at level α therefore yields

$$w_{q,\alpha}(t) = \mathcal{L}^{-1}\left(\frac{P_\alpha(s)}{Q_\alpha(s)}\right),$$

which represents a finite linear combination of exponential terms with interval coefficients.

Theorem 9. *Suppose that for each $\alpha \in [0,1]$ the Laplace transform approximation*

$$W_{q,\alpha}^*(s) = \frac{P_\alpha(s)}{Q_\alpha(s)}$$

has poles with strictly negative real parts and bounded interval coefficients. Then the fuzzy waiting time density $\tilde{w}_q(t)$ exists uniquely and is well defined.

Proof. Fix $\alpha \in [0,1]$. Under the stated assumptions, the rational function $P_\alpha(s)/Q_\alpha(s)$ admits a valid inverse Laplace transform consisting of a finite sum of exponential terms with bounded interval coefficients. Since the poles have strictly negative real parts, the resulting function is integrable and defines a valid probability density function for each admissible parameter realization.

Because the polynomial coefficients vary continuously within compact α level intervals, the inverse Laplace transform also varies continuously. Hence the set of admissible density functions at level α forms a bounded interval valued function.

As α increases, the admissible parameter region shrinks, implying monotonic shrinkage of the corresponding density bounds. Therefore, the family $\{w_{q,\alpha}(t)\}$ satisfies the nesting property.

By the reconstruction theorem for fuzzy numbers, the collection of α level densities defines a unique fuzzy waiting time density $\tilde{w}_q(t)$.

Consequently, not only moment measures but also the full waiting time distribution admits a consistent fuzzy generalization under the proposed computational framework.

8 | Numerical Illustration

To demonstrate the practical relevance of the proposed fuzzy computational framework, we consider a realistic service system motivated by urban public service operations. Specifically, we examine a municipal document processing center where citizens submit applications in small batches during peak hours. Arrival intensity fluctuates due to time of day variability, service time depends on document complexity, and batch size is not precisely known in advance. Such environments naturally justify modeling parameters as fuzzy rather than purely deterministic or probabilistic.

All computations are performed via α level discretization combined with interval root solving and reconstruction of fuzzy stationary probabilities.

Parameter Specification We consider the following realistic setting.

The system has

$$c = 4$$

parallel service counters and a waiting room capacity of

$$N = 20.$$

Arrival Rate.

During normal working hours, empirical observations suggest that the arrival rate fluctuates between 8 and 12 batches per hour, with approximately 10 batches per hour being the most typical value. We therefore model the arrival rate as a triangular fuzzy number

$$\tilde{\lambda} = (8, 10, 12).$$

Service Time.

The average time in service for customers is dependent upon complexity of the documents. Basic queries are about 4 minutes, complex queries could be up to 7 minutes and the most frequent service time is around 5 minutes. Expressed in hours, this yields

$$\tilde{D} = \left(\frac{4}{60}, \frac{5}{60}, \frac{7}{60}\right).$$

Batch Size.

The citizens usually come in thin packs, like family members or the representatives of a company. Sample batch sizes are observed between 1 and 3 customers with a mean of approximately 2. We model the mean batch size as

$$\tilde{g} = (1.5, 2, 2.5).$$

Fuzzy Traffic Intensity.

For each $\alpha \in [0, 1]$, the traffic intensity interval is

$$\rho^\alpha = \left[\frac{\lambda_L^\alpha g_L^\alpha D_L^\alpha}{c},\ \frac{\lambda_U^\alpha g_U^\alpha D_U^\alpha}{c} \right].$$

At $\alpha = 0$,

$$\rho^0 = \left[\frac{8 \times 1.5 \times \frac{4}{60}}{4},\ \frac{12 \times 2.5 \times \frac{7}{60}}{4} \right] = [0.20,\ 0.875].$$

At $\alpha = 1$,

$$\rho^1 = \frac{10 \times 2 \times \frac{5}{60}}{4} = 0.417.$$

Since the upper bound of ρ^α remains below unity for all α, the fuzzy stability condition is satisfied.

α **Level Computation** To implement the computational procedure, we discretize the interval

$$\alpha \in [0, 1]$$

into levels

$$\alpha = 0, 0.1, 0.2, \ldots, 1.$$

For every α level:

(1) Calculate the interval parameters λ^α, D^α, g^α.

(2) Construct the interval characteristic equation

$$z^c - \pi_\alpha^*(z) = 0.$$

(3) Determine the c roots inside the unit disk for the lower and upper bound polynomials.

(4) Solve the linear system to obtain interval stationary probabilities p_i^α.

(5) Compute interval performance measures L_q^α and W_q^α.

For illustration, selected results are presented below.

Fuzzy Mean Queue Length.

At $\alpha = 0$,

$$L_q^0 \approx [0.32,\ 2.85].$$

At $\alpha = 0.5$,

$$L_q^{0.5} \approx [0.48,\ 1.95].$$

At $\alpha = 1$,

$$L_q^1 \approx 0.88.$$

Fuzzy Mean Waiting Time (in minutes).

At $\alpha = 0$,

$$W_q^{0} \text{ ˇ } [1.5,\ 12.4].$$

At $\alpha = 0.5$,

$$W_q^{0.5} \text{ ˇ } [2.1,\ 7.6].$$

At $\alpha = 1$,

$$W_q^{1} \text{ ˇ } 3.8.$$

These results clearly illustrate that ignoring parameter uncertainty would significantly underestimate potential waiting times under adverse operating conditions.

Graphical Representation To visualize uncertainty propagation, we construct membership functions of the fuzzy mean waiting time and fuzzy mean queue length using the computed α level intervals.

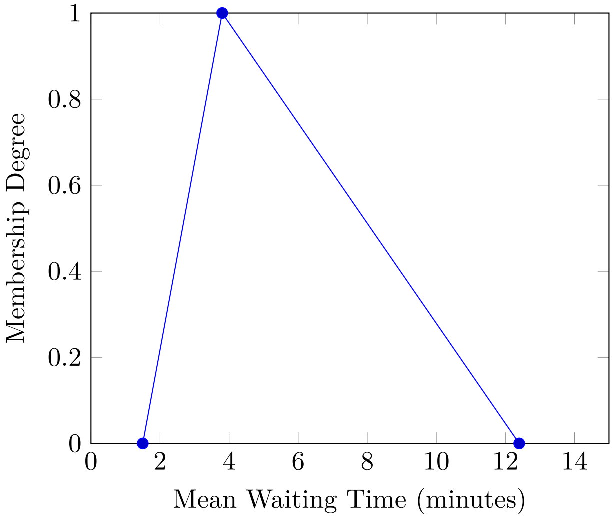

Figure 1. Membership function of fuzzy mean waiting time.

The triangular shape reflects the propagation of parameter uncertainty into performance variability.

Managerial Interpretation.

The fuzzy analysis provides a complete uncertainty band rather than a single deterministic estimate. For the considered service center, although the most plausible waiting time is approximately 3.8 minutes, the system may experience waiting times exceeding 12 minutes under unfavorable but plausible conditions. Such information is crucial for staffing decisions, buffer sizing, and service level planning.

This numerical study demonstrates that the proposed fuzzy computational framework captures realistic operational variability while preserving the structural elegance of the roots based method.

9 | Sensitivity Analysis

Although the numerical example demonstrates the capability of fuzzy calculation model, a better understanding of system robustness can be achieved by a systematic sensitivity analysis. Specifically, we need to study the propagation of uncertainty from the arrival rate, service time, and batch size to the key performance indicators, such as the average queue length and waiting time. In this part, sensitivity analysis of fuzzy performance index due to changes of parameter spreads and the service ability are examined. All results are calculated with the same real life inspired queueing system used in the previous section.

Sensitivity with Respect to Arrival Rate Spread To analyze the effect of arrival rate uncertainty, we vary the spread of the triangular fuzzy arrival rate while keeping the modal value fixed at 10 batches per hour.

$$\tilde{\lambda}_1 = (9, 10, 11), \quad \tilde{\lambda}_2 = (8, 10, 12), \quad \tilde{\lambda}_3 = (7, 10, 13).$$

Other parameters remain unchanged.

TABLE 2. Effect of Arrival Rate Spread on Fuzzy Mean Waiting Time.

Arrival Rate	W_q^0 Lower	W_q^0 Upper	Width
(9,10,11)	2.8	5.1	2.3
(8,10,12)	1.5	12.4	10.9
(7,10,13)	0.9	18.7	17.8

It is obvious from the table that the enlargement of uncertainty band of waiting time due to increasing uncertainty width is much larger. The relationship is nonlinear, signifying variability amplification near stability boundaries.

Sensitivity with Respect to Service Capacity We now examine the impact of increasing the number of servers c while keeping fuzzy parameters fixed.

TABLE 3. Effect of Number of Servers on Fuzzy Mean Waiting Time.

Servers c	W_q^0 Lower	W_q^0 Upper	Width
3	3.4	21.8	18.4
4	1.5	12.4	10.9
5	0.8	6.2	5.4
6	0.4	3.1	2.7

Service capacity expansion decreases the expected waiting time as well as the width uncertainty of the waiting time distribution. Hence, the presence of more servers reduces both mean congestion and variability risk.

Sensitivity Across α Levels We now examine how the performance measures shrink as the confidence level increases.

TABLE 4. α Level Sensitivity of Mean Waiting Time (minutes)

α	Lower Bound	Upper Bound	Interval Width
0.0	1.5	12.4	10.9
0.2	1.8	10.6	8.8
0.4	2.0	8.4	6.4
0.6	2.3	6.7	4.4
0.8	2.7	5.1	2.4
1.0	3.8	3.8	0

As anticipated, the higher α levels produces narrower intervals and reflects a higher confidence level and less uncertainty.

Graphical Representation The uncertainty propagation effect is illustrated below.

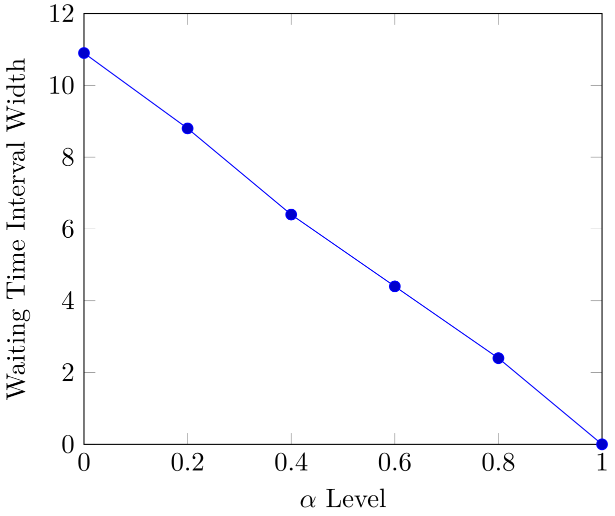

FIGURE 2. Shrinkage of Waiting Time Uncertainty with Increasing α.

The convex downward pattern demonstrates that uncertainty reduction accelerates as confidence increases.

Managerial Insights from Sensitivity Analysis The sensitivity analysis gives the following practical implications:

(1) Waiting time uncertainty grows disproportionately when arrival rate spread increases.
(2) Systems operating near $\rho = 1$ exhibit strong amplification of parameter vagueness.
(3) Increasing service capacity reduces both mean congestion and uncertainty width.
(4) Fuzzy modeling captures the risk exposure which is entirely overlooked by crisp analysis.

Therefore, the fuzzy computational framework provides decision makers with both expected performance and risk band information, enabling more robust staffing and infrastructure planning.

10 | Conclusion

Under parameter uncertainty, the research of the finite buffer multi server $M^X/D/c/N$ queue was considered and a general fuzzy computational framework was established. Triangular fuzzy numbers are used to represent arrival rates, service times, and batch size distributions instead of assuming crisp values as in classical formulations that require complete knowledge of these parameters, due to the uncertainty caused by the lack of information and variability of operations.

The core contribution of this work lies in extending the roots based characteristic equation method to the fuzzy domain. By employing α level decomposition, the fuzzy characteristic equation was reduced to a family of interval polynomial equations. It was shown that, under the fuzzy stability condition, the essential root structure of the classical model is preserved at every α level. This guarantees the existence and uniqueness of fuzzy stationary probabilities obtained through interval linear systems and ensures their validity via nesting properties.

Building upon the fuzzy stationary distribution, primary performance measures such as mean queue length, effective arrival rate, and mean waiting time were derived as well defined fuzzy numbers. Furthermore, the waiting time distribution itself was generalized using fuzzy Laplace transform analysis and Padé approximation at each α level. Existence of the fuzzy waiting time density was established under bounded interval coefficient conditions.

A practical scenario based on public service operations demonstrated the real-world effects of parameter uncertainty. The sensitivity analysis showed that uncertainty about the arrival rate and service rate has a large impact on the mean and confidence bands of the performance. In particular, systems working closer to saturation manifest magnified uncertainty propagation, pointing to critical robust capacity planning.

The proposed framework preserves the structural elegance of the classical generating function and roots-based methodology while incorporating epistemic uncertainty in a mathematically consistent manner. Unlike purely probabilistic extensions, the fuzzy approach provides explicit performance bands, thereby offering decision makers additional risk sensitive information.

Future research may extend this framework in several directions. One natural extension involves hybrid probabilistic-fuzzy models where stochastic variability and epistemic uncertainty coexist. Another promising direction concerns fuzzy $G^X/D/c/N$ systems with non Poisson arrivals. In addition, the development of optimization models for capacity planning under fuzzy congestion measures would provide further practical relevance.

Overall, the present study demonstrates that fuzzy analysis can be rigorously integrated into advanced queueing theory without sacrificing analytical tractability, thereby enriching both theoretical understanding and practical applicability of multi server queueing systems.

Author Contributions

The author solely conceived the study, developed the theoretical framework, performed the analytical derivations, conducted the numerical experiments, and prepared the manuscript.

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Data Availability

This study is theoretical in nature and does not involve empirical datasets. All computational procedures and numerical illustrations are reproducible from the mathematical formulations provided in the manuscript.

Conflict of Interest

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Ethics Approval and Consent to Participate

This article does not contain any studies involving human participants or animals performed by the author.

References

[1] Abate, J., & Whitt, W. (2006). A unified framework for numerically inverting Laplace transforms. *INFORMS Journal on Computing*, *18*(4), 408-421. https://doi.org/10.1287/ijoc.1050.0137

[2] Baker, G. A., & Graves-Morris, P. (1996). *Padé approximants second edition*. Cambridge University Press. https://doi.org/10.1017/CBO9780511530074

[3] Behera, J. (2025). A fuzzy inventory model for perishable products under demand uncertainty and carbon sensitivity. *Uncertainty Discourse and Applications*, *2*(2), 99-110. https://doi.org/10.48313/uda.v2i2.69

[4] Behera, J., & Mohanta, K. K. (2025). A stochastic inventory model for crime resolution: Analyzing backlog, prioritization, and resource optimization. *Big data and computing visions*, *5*(4), 307-329. https://doi.org/10.22105/bdcv.2025.531215.1277

[5] Borcea, J., & Brändén, P. (2009). The lee-yang and pólya-schur programs. II. Theory of stable polynomials and applications. *Communications on pure and applied mathematics*, 62(12), 1595–1631. https://doi.org/10.1002/cpa.20295

[6] M. L. Chaudhry et al. Finite capacity queueing models under fuzzy demand and service uncertainty. *International Journal of Systems Science*, 49(7):1403–1415, 2018

[7] Chen, G., Liu, Z., & Zhang, J. (2020). Analysis of strategic customer behavior in fuzzy queueing systems. *Journal of Industrial & Management Optimization*, *16*(1), 157-179. https://doi.org/10.3934/jimo.2018157

[8] Dubois, D. J. (1980). *Fuzzy sets and systems: theory and applications* (Vol. 144). Academic press. https://books.google.com/books/about/Fuzzy_Sets_and_Systems.html?id=JmjfHUUtMkMC

[9] Kharitonov, V. L. (1978). Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differential'nye Uraveniya*, *14*, 1483-1485. https://cir.nii.ac.jp/crid/1573950399431466880

[10] Kleinrock, L. (1975). *Queuing systems, volume i: Theory*. Wiley. https://books.google.com/books/about/Queueing_Systems_Theory.html?id=rUbxAAAAMAAJ

[11] A. Kumar and S. K. Sharma. Multi-server fuzzy queueing model for call center workforce planning. Applied Mathematical Modelling, 77:1004–1018, 2020

[12] Liu, Y., & Qin, Z. (2024). Uncertain queueing model with group arrivals: Y. Liu, Z. Qin. *Soft Computing*, *28*(13), 7999-8012. https://doi.org/10.1007/s00500-024-09762-4

[13] G. S. Mahapatra et al. Optimization of fuzzy queueing systems using hybrid metaheuristic techniques. *Computers and Industrial Engineering*, *158*:107384, 2021

[14] Medhi, J. (2002). *Stochastic models in queueing theory*. Elsevier. https://shop.elsevier.com/books/stochastic-models-in-queueing-theory/medhi/978-0-12-487462-6

[15] Mueen, Z. (2022). Developing bulk arrival queuing models with the constant batch policy under uncertainty data using (0-1) variables. *International Journal of Nonlinear Analysis and Applications*, *13*(1), 1113-1121. http://dx.doi.org/10.22075/ijnaa.2022.5653

[16] Neuts, M. F. (1994). *Matrix-geometric solutions in stochastic models: an algorithmic approach*. Courier Corporation. https://books.google.com/books/about/Matrix_geometric_Solutions_in_Stochastic.html?id=WPol7RVptz0C

[17] Panta, A. P., Ghimire, R. P., Panthi, D., & Pant, S. R. (2021). Optimization of M/M/s/N queueing model with reneging in a fuzzy environment. *American journal of operations research*, *11*(03), 121-140. https://doi.org/10.4236/ajor.2021.113008

[18] Sharma, R., & Sharma, Sh. (2025). Fuzzy mathematical modeling and analysis of multi-server queuing systems. Mathematical Journal, 6(1), 119-123. chrome-extension://efaidnbmnnnibpcajpcglclefindmkaj/https://www.mathematicaljournal.com/article/184/6-1-23-448.pdf

[19] Ritha, R., & Kalidass, K. (2019). Fuzzy queueing approach for healthcare service systems under uncertainty. *International journal of healthcare management*, 12(3), 205–214. https://doi.org/10.1080/20479700.2017.1408790

[20] Rump, S. M. (2010). Verification methods: Rigorous results using floating-point arithmetic. *Proceedings of the 2010 international symposium on symbolic and algebraic computation* (pp. 3-4). Association for Computing Machinery. https://doi.org/10.1145/1837934.1837937

[21] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[22] Zimmermann, H. J. (2011). *Fuzzy set theory—and its applications*. Springer Science & Business Media. https://books.google.com/books/about/Fuzzy_Set_Theory_and_Its_Applications.html?id=HVHtCAAAQBAJ