Uncertainty Discourse and Applications

www.uda.reapress.com

Uncert. Disc. Appl. Vol. 3, No. 1 (2026) 14–32.

Paper Type: Original Article

On the Foundation of Solving a First-Order Weak Fuzzy Complex Initial Value Problem (WFC-IVP)

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Citation:

Received: 10 August 2025 *Revised: 06 November 2025* *Accepted: 21 December 2025*	Razouk, L. (2026). On the foundation of solving a first-order weak fuzzy complex initial value problem (WFC-IVP). *Uncertainty Discourse and Applications, 3*(1), 14-32.

Abstract

The aim of this paper is to introduce for the first time the basic concept of the first-order Weak Fuzzy Complex-Initial Value Problems (WFC-IVPs). We find, using a special isomorphic transformation function, that solving a WFC-IVP is equivalent to solving two classical real IVPs with respect to their own real variables. Thus, we study "the existence and uniqueness", "the stability", and "the well-posedness" associated with the WFC-IVPs in terms of definitions, lemmas, and theorems. Then, we get the approximate solutions of a WFC-IVP by stable and convergent numerical methods. One of the most famous and simple methods to solve IVPs is Euler's method, which is discussed for well-posed WFC-IVPs. However, we focus on a stable linear model WFC-IVP with real coefficients and real initial values, and we further investigate the properties of the results. Additionally, we present an example with tables and diagrams of its numerical solutions and absolute errors by Python to clarify how Euler's algorithm works.

Keywords: Weak fuzzy complex set, Weak fuzzy complex functions, Initial value problem, Euler's method.

1 | Introduction

Over centuries, the extensions of number sets have been created out of necessity often to solve specific problems. Researchers have been very interested in expanding the set of real numbers, where the developed sets have been defined as two-dimensional generalizations of real numbers such as:

Complex numbers: $\{u + v\,i;\ u, v \in R;\ i^2 = -1\}$, Dual numbers: $\{u + vi;\ u, v \in R;\ i^2 = 0\}$, Neutrosophic numbers: $\{u + vi;\ u, v \in R;\ i^2 = i\}$, and Split-Complex numbers: $\{u + v\,i;\ u, v \in R;\ i^2 = 1\}$.

Recently, in a similar way to build sets of Complex, Dual, Neutrosophic, and Split-Complex numbers, the Weak Fuzzy Complex (WFC) set $\{u + v\,J;\ u, v \in R; J^2 = t \in [0, 1]\}$ was born in 2023 [1]. It is a new generalization of classical real numbers set with fuzzy operators $J \notin R$. Hence, they introduced the solutions of linear and quadratic WFC equations [1], the WFC vector space [2], the inner products defined over this vector space [3], and linear systems by applying WFC matrices [4].

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https://doi.org/10.48313/uda.vi.81

The linear Diophantine equation in two WFC variables was studied in [5]. Also, the non-linear WFC Diophantine equations have been solved in [6], [7] with the concepts of the WFC and anti-WFC Pythagoras triples and Pythagoras quadruples. Additionally, Abobala [8] defined some main notions in number theory, like division, ordering, and units in the set of WFC integers.

One important application of the WFC numbers is 'A-Curves' which display the geometrical characterization of the solutions for some vectorial equations defined by Euclidean norms [9]. Also, many new semi-module isomorphisms gave a full classification of A-Curves related to the WFC ring [10].

In computer [11], Python introduced WFC numbers and their main arithmetic operations.

In addition, Razouk et al. [12] created a special isomorphism transformation function that works between the WFC set and the Real number set, and they defined the foundation of WFC functions. Thus, using this transformation function, Edduweh et al. [13] provided the basic concepts of WFC Ordinary Differential Equations (WFC-ODEs) and found the solutions of some types of first-order first-degree separable, exact, and linear WFC-ODEs.

While Hatamleh [14] studied the novel topological space generated by WFC intervals based on the partial ordered ring structure of WFC numbers.

Otherwise, we know that an Initial Value Problems (IVPs) play an essential role in modeling World phenomena. Thus, the IVPs have predictive capability which is vital in many applications, including weather forecasting [15], financial modeling [16], and control systems [17].

That motivates us to study an important research gap about the IVPs for WFC-ODEs. In this work, we consider a numerical method for solving a stable model of first-order first-degree WFC-IVP.

Basic notions about WFC elements are mentioned in Section 2, where we introduce the Mean Value Theorem in F_J. Accordingly, we define the standard form of first-order WFC-IVP in Section 3, with the related fundamental issues: existence, uniqueness, stability, and well-posedness. Then, we show the methodology of the numerical techniques in Section 4 to obtain approximate solutions to the well-posed first-order WFC-IVP. In Section 5, we address Taylor series Methods with the numerical errors that can be formed. We explain "Euler's method" which is considered the basis of methods for solving the IVP, and infer its properties for our WFC-IVP. A linear stable model problem will be considered in Section 6, where we apply Euler's algorithm to it and prove the similar results between the original IVP in F_J and its two related IVPs in R. After that, a simple example appears with tables, and diagrams of solutions and errors obtained using Python language (the instructions executed by Python are included in Appendix) to illustrate the validity of our approach.

2 | Preliminaries

The subsequent definitions will help clarify the various elements at play in our analysis, thereby facilitating a deeper understanding of the results we present for solving IVPs in a WFC variable [10], [18].

Definition 1. The set of WFC numbers was defined as follows:

$$F_J = \{x_0 + x_1 J \,;\, x_0, x_1 \in R, J^2 = t \in [0,1]\},$$

where 'J' is the WFC operator ($J \notin R$).

To simplify, we transform from elements in F_J to real elements using a special transformation function, and then go back to F_J by its inverse [12].

Definition 2. Let φ be the transformation function from F_J to $R \times R$, which we define as follows:

$$\varphi: F_J \mapsto R \times R,$$

$$\varphi(x_0 + x_1 J) = (x_0 + x_1(-\sqrt{t}), x_0 + x_1(+\sqrt{t})) = (x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}),$$

where $J^2 = t \in [0,1] \Rightarrow J = \pm\sqrt{t}$, and $x_0, x_1, y_0, y_1 \in R$ (This map is an isomorphism).

Definition 3. Let $\varphi: F_J \mapsto R \times R$ such that $\varphi(X) = (a,b)$, the inverse function of φ is defined as follows:

$$\varphi^{-1}: R \times R \mapsto F_J,$$

$$\varphi^{-1}(a,b) = \frac{1}{2}[a+b] + \frac{1}{2\sqrt{t}}J[b-a],$$

Definition 4. Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in F_J$, we say that $X \le Y$, if and only if:

$$\begin{cases} x_0 - x_1\sqrt{t} \le y_0 - y_1\sqrt{t}, \\ x_0 + x_1\sqrt{t} \le y_0 + y_1\sqrt{t}. \end{cases}$$

Definition 5. Let $A = a_0 + a_1 J$, $B = b_0 + b_1 J \in F_J$, we define the interval $[A,B]$ if and only if $A \le B$, according to the definition of the partial order relation ($\le$).

I. If $A \nleq B$, then $[A,B] = \phi$.

II. We can understand $[A,B]$ as follows:

$$[A,B] = \{C = c_0 + c_1 J \in F_J;\ A \le C \le B\}.$$

Definition 6. Let $f: F_J \mapsto F_J$ be a WFC function in one variable, where

$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$; $f_1, f_2: R \mapsto R$, then we say:

I. f is continuous on F_J if and only if f_1, f_2 are continuous on R.

II. f is differentiable on F_J if and only if f_1, f_2 are differentiable on R, with respect to their own variables.

III. f is integrable on F_J if and only if f_1, f_2 are integrable on R.

Definition 7. Let $f: F_J \mapsto F_J$ be a differentiable/integrable function on F_J. We define

I. $f'(X) = \varphi^{-1}\left(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})\right)$.

II. $\int f(X).dX = \varphi^{-1}\left(\int f_1.d(x_0 - x_1\sqrt{t}), \int f_2.d(x_0 + x_1\sqrt{t})\right)$.

Definition 8. The Taylor series of a WFC function $f(X)$ that is infinitely differentiable at a WFC number $X_0 = a + bJ$ is the series:

$$f(X) = \sum_{n=0}^{\infty} f^{(n)}(X_0)\frac{(X - X_0)^n}{n!} \quad ; X, X_0 \in F_J,$$

where $n \in N$, $f^{(n)}(X_0)$ is denotes the n-th derivative of f evaluated at the point X_0.

Definition 9. We consider the explicit WFC first-order differential equation

$$Y' = F(X,Y),$$

where

$$Y = y_0 + y_1 J = \varphi^{-1}(Y_0, Y_1) = \varphi^{-1}\left(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}\right) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})),$$

$$Y' = \varphi^{-1}(Y_0', Y_1') = \varphi^{-1}\left(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})\right) = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2),$$

$$Y_0 = y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t}) = \int \mathcal{F}_1 d(x_0 - x_1\sqrt{t}),$$

$$Y_1 = y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t}) = \int \mathcal{F}_2 d(x_0 + x_1\sqrt{t}),$$

$$X = \varphi^{-1}(X_0, X_1) = \varphi^{-1}(x_0 - x_1\sqrt{t},\ x_0 + x_1\sqrt{t}),$$

$$X_0 = x_0 - x_1\sqrt{t} \in I_1 \subseteq R, X_1 = x_0 + x_1\sqrt{t} \in I_2 \subseteq R, I = \varphi^{-1}(I_1 \times I_2) \subseteq F_J,$$

where

$$Y = y_0 + y_1 J = \varphi^{-1}(Y_0, Y_1) = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})),$$

$$Y' = \varphi^{-1}(Y_0', Y_1') = \varphi^{-1}(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})) = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2),$$

$$Y_0 = y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t}) = \int \mathcal{F}_1 d(x_0 - x_1\sqrt{t}),$$

$$Y_1 = y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t}) = \int \mathcal{F}_2 d(x_0 + x_1\sqrt{t}),$$

$$X = \varphi^{-1}(X_0, X_1) = \varphi^{-1}(x_0 - x_1\sqrt{t},\ x_0 + x_1\sqrt{t}),$$

$$X_0 = x_0 - x_1\sqrt{t} \in I_1 \subseteq R, X_1 = x_0 + x_1\sqrt{t} \in I_2 \subseteq R, I = \varphi^{-1}(I_1 \times I_2) \subseteq F_J.$$

Remark 1. The Lipschitz condition equation is equivalent to the two Lipschitz conditions (due to *Definition 4*).

$$\begin{cases} |\mathcal{F}_1(X_0, Z_0) - \mathcal{F}_1(X_0, S_0)| \leq L_0|Z_0 - S_0|, \\ |\mathcal{F}_2(X_1, Z_1) - \mathcal{F}_2(X_1, S_1)| \leq L_1|Z_1 - S_1|, \end{cases}$$

where (X_0, Z_0) and (X_0, S_0) are in $D_1 = \{(X_0, Y_0);\ X_0 \in I_1 = [A_0, B_0] \subseteq R, -\infty < Y_0 < +\infty\}$,

(X_1, Z_1) and (X_1, S_1) are in $D_2 = \{(X_1, Y_1);\ X_1 \in I_2 = [A_1, B_1] \subseteq R, -\infty < Y_1 < +\infty\}$, and

$L_0, L_1 > 0$ are Lipschitz constants for $\mathcal{F}_1, \mathcal{F}_2$, respectively.

Theorem 1 (The Mean Value Theorem on F_J). If f is a WFC continuous function on an interval $[A, B] \subseteq F_J$ and differentiable on (A, B), then a WFC number C in (A, B) exists with

$$f'^{(C)} = \frac{f(B) - f(A)}{B - A},$$

where

$$Y = \varphi^{-1}(Y_0, Y_1), \quad Y_0 = y_0 - y_1\sqrt{t},\ Y_1 = y_0 + y_1\sqrt{t},$$

$$X = \varphi^{-1}(X_0, X_1), \quad X_0 = x_0 - x_1\sqrt{t},\ X_1 = x_0 + x_1\sqrt{t},$$

$$A = \varphi^{-1}(A_0, A_1), \quad A_0 = a_0 - a_1\sqrt{t},\ A_1 = a_0 + a_1\sqrt{t},$$

$$B = \varphi^{-1}(B_0, B_1), \quad B_0 = b_0 - b_1\sqrt{t},\ B_1 = b_0 + b_1\sqrt{t},$$

$$C = \varphi^{-1}(C_0, C_1), \quad C_0 = c_0 - c_1\sqrt{t},\ C_1 = c_0 + c_1\sqrt{t}.$$

Proof: f is a WFC function, then we have

$$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})); f_1, f_2: R \mapsto R,$$

when f is continuous on $[A, B] \subseteq F_J$, then f_1, f_2 are continuous on $[A_0, B_0], [A_1, B_1] \subseteq R$, respectively.

Also, f is differentiable on $(A, B) \subseteq F_J$, then f_1, f_2 are differentiable on $(A_0, B_0), (A_1, B_1) \subseteq R$, with respect to their own variables.

Depending on the mean value theorem in R [9] numbers C_1, C_2 in $(A_0, B_0), (A_1, B_1)$ exist with:

$$f_1{}'(C_1) = \frac{f_1(B_0) - f_1(A_0)}{B_0 - A_0} \text{ and } f_2^{\prime(C_2)} = \frac{f_2(B_1) - f_2(A_1)}{B_1 - A_1}.$$

And we know that:

$$\begin{aligned} f'(C) = \varphi^{-1}\big(f_1'(C_1), f_2'(C_2)\big) &= \varphi^{-1}\left(\frac{f_1(B_0) - f_1(A_0)}{B_0 - A_0}, \frac{f_2(B_1) - f_2(A_1)}{B_1 - A_1}\right), \\ &= \varphi^{-1}\left(\frac{(f_1(B_0), f_2(B_1)) - (f_1(A_0), f_2(A_1))}{(B_0, B_1) - (A_0, A_1)}\right), \\ &= \frac{f(B) - f(A)}{B - A}. \end{aligned}$$

3 | The Initial-Value Problem for a first-Order Differential Equation in F_J

The IVP for first-order differential equations serves as a crucial tool in understanding dynamic systems across various disciplines. We first establish key definitions that will serve as the foundation for our subsequent discussions.

Definition 12. The WFC- IVP (WFC-Cauchy problem) consists of finding the solution of a WFC-ODE given suitable initial conditions. The standard form of the first-order WFC-ODE that has been adopted is

$$\begin{cases} Y' = \mathcal{F}(X, Y), \\ Y(A) = C \quad \text{(the initial condition)}, \end{cases} \qquad \begin{cases} dY = \mathcal{F}(X, Y)dX, \\ Y(A) = C, \end{cases} \tag{1}$$

where a WFC function $\mathcal{F}(X, Y(X))$ defined on a set D, and a fixed WFC value $(A, C) \in D$ be given. A function $Y(X)$ is sought that is differentiable in an interval I with $A \in I$, and

$$D = \{(X, Y); X \in I;\ I = [A, B] \subseteq F_J, -\infty < Y_0, Y_1 < +\infty\} \subseteq F_J \times F_J;\ D = \varphi^{-1}(D_1, D_2),$$

$$D_1 = \{(X_0, Y_0);\ X_0 \in I_1 = [A_0, B_0] \subseteq R, -\infty < Y_0 < +\infty\},$$

$$D_2 = \{(X_1, Y_1);\ X_1 \in I_2 = [A_1, B_1] \subseteq R, -\infty < Y_1 < +\infty\}$$

$$X = \varphi^{-1}(X_0, X_1), \qquad X_0 = x_0 - x_1\sqrt{t},\ X_1 = x_0 + x_1\sqrt{t},$$

$$Y = \varphi^{-1}(Y_0, Y_1), \qquad Y_0 = y_0 - y_1\sqrt{t},\ Y_1 = y_0 + y_1\sqrt{t},$$

$$A = \varphi^{-1}(A_0, A_1), \qquad A_0 = a_0 - a_1\sqrt{t},\ A_1 = a_0 + a_1\sqrt{t},$$

$$B = \varphi^{-1}(B_0, B_1), \qquad B_0 = b_0 - b_1\sqrt{t},\ B_1 = b_0 + b_1\sqrt{t},$$

$$C = \varphi^{-1}(C_0, C_1), \qquad C_0 = c_0 - c_1\sqrt{t},\ C_1 = c_0 + c_1\sqrt{t}.$$

Definition 13. Suppose that $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F} = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2)$ are differentiable in $I_1 = [A_0, B_0], I_2 = [A_1, B_1]$ and $I = [A, B] \subseteq F_J$ with respect to their own variables X_0, X_1 and X, respectively. A function $Y(X): I \to F_J$ is called the solution to a WFC-IVP *Eq. (1)* and given as follows:

$$Y = \varphi^{-1}\big(Y_0(X_0), Y_1(X_1)\big) = \frac{1}{2}\big(Y_0(X_0) + Y_1(X_1)\big) + \frac{1}{2\sqrt{t}}J\big(Y_1(X_1) - Y_0(X_0)\big),$$

where $Y_0(X_0)$ and $Y_1(X_1)$ are the corresponding solutions of *Eq. (2)* and *Eq. (3)*, respectively

$$\begin{cases} dY_0 = \mathcal{F}_1(X_0, Y_0)dX_0 \\ Y_0(A_0) = C_0 \end{cases}; A_0 \leq X_0 \leq B_0, \quad \textbf{(2)}$$

$$\begin{cases} dY_1 = \mathcal{F}_2(X_1, Y_1)dX_1 \\ Y_1(A_1) = C_1 \end{cases}; A_1 \leq X_1 \leq B_1. \quad \textbf{(3)}$$

Proof: using the function φ for *Eq. (1)*, we get

$$\begin{cases} \varphi^{-1}(dY_0, dY_1) = \varphi^{-1}\big((\mathcal{F}_1(X_0, Y_0), \mathcal{F}_2(X_1, Y_1))(dX_0, dX_1)\big), \\ \varphi^{-1}\big(Y_0(A_0), Y_1(A_1)\big) = \varphi^{-1}(C_0, C_1), \end{cases}$$

$$\Rightarrow \begin{cases} \begin{cases} dY_0 = \mathcal{F}_1(X_0, Y_0)dX_0 \\ Y_0(A_0) = C_0 \end{cases}; A_0 \leq X_0 \leq B_0, \\ \begin{cases} dY_1 = \mathcal{F}_2(X_1, Y_1)dX_1 \\ Y_1(A_1) = C_1 \end{cases}; A_1 \leq X_1 \leq B_1, \end{cases}$$

So, we can say that finding the solution for the WFC-IVP *Eq. (1)* is equivalent to solve the two related Cauchy problems in R, *Eq. (2)* and *Eq. (3)* by φ, and the solution has the form

$$Y = Y(X) = \varphi^{-1}\big(Y_0(X_0), Y_1(X_1)\big) = \frac{1}{2}\big(Y_0(X_0) + Y_1(X_1)\big) + \frac{1}{2\sqrt{t}}J\big(Y_1(X_1) - Y_0(X_0)\big).$$

From another side, to verify that the solution $Y(X)$ satisfies *Eq. (1)* ($Y(X)$ satisfies the differential equation and the initial condition)

I. $Y'(X) = \varphi^{-1}(Y_0'(X_0), Y_1'(X_1)) = \frac{1}{2}(Y_0'(X_0) + Y_1'(X_1)) + \frac{1}{2\sqrt{t}}J(Y_1'(X_1) - Y_0'(X_0)),$

$$= \frac{1}{2}\big(\mathcal{F}_1(X_0, Y_0) + \mathcal{F}_2(X_1, Y_1)\big) + \frac{1}{2\sqrt{t}}J\big(\mathcal{F}_2(X_1, Y_1) - \mathcal{F}_1(X_0, Y_0)\big),$$

$$= \varphi^{-1}\big(\mathcal{F}_1(X_0, Y_0), \mathcal{F}_2(X_1, Y_1)\big) = \mathcal{F}(X, Y).$$

II. $Y(A) = \varphi^{-1}\big(Y_0(A_0), Y_1(A_1)\big) = \frac{1}{2}\big(Y_0(A_0) + Y_1(A_1)\big) + \frac{1}{2\sqrt{t}}J\big(Y_1(A_1) - Y_0(A_0)\big),$

$$= \frac{1}{2}(C_0 + C_1) + \frac{1}{2\sqrt{t}}J(C_1 - C_0) = \varphi^{-1}(C_0, C_1) = C.$$

In advance of trying to solve a WFC-IVP, it is desirable to know whether a solution exists, and when it does, whether it is the only solution to the problem.

3.1 | The Existence and Uniqueness

Theorem 2 (The Existence and Uniqueness theorem). Suppose $\mathcal{F}(X, Y)$ is continuous on $D = \{(X, Y); X \in I;\ I = [A, B] \subseteq F_J, -\infty < Y_0, Y_1 < +\infty\}$. If $\mathcal{F}$ satisfies a Lipschitz condition on D in the variable Y, then the WFC-IVP *Eq. (1)* has a unique solution $Y(X)$ for $X \in I$.

Proof: we know that solving WFC-IVP *Eq. (1)* is equivalent to solve *Eq. (2)* and *Eq. (3)* in R. And

I. Since $\mathcal{F}(X, Y)$ is continuous on D if and only if $\mathcal{F}_1, \mathcal{F}_2$ are continuous on $D_1, D_2 \subseteq$ R.

II. Since $\mathcal{F}(X, Y)$ satisfies a Lipschitz condition on D if and only if $\mathcal{F}_1, \mathcal{F}_2$ satisfy Lipschitz conditions on D_1, D_2.

Then each IVP *Eq. (2)* and *Eq. (3)* has a unique solution $Y_0(X_0), Y_1(X_1)$, respectively (The existence and uniqueness theorem in R [9], [19]).

It means that $Y(X)$-the solution of *Eq. (1)*, is a unique structure of two unique solutions in R:

$$Y(X) = \varphi^{-1}(Y_0(X_0), Y_1(X_1)) = \frac{1}{2}\big(Y_0(X_0) + Y_1(X_1)\big) + \frac{1}{2\sqrt{t}}J\big(Y_1(X_1) - Y_0(X_0)\big).$$

Consequently, we can assume the following lemma.

Lemma 1. "The existence of the unique solution $Y(X) \in F_J$ of *Eq. (1)*" is equivalent to "the existence of the two unique solutions $Y_0(X_0)$ and $Y_1(X_1) \in R$ of *Eq. (2)* and *Eq. (3)*, respectively"

Example 1.

$$\begin{cases} Y' = 1 + X\sin(XY), \\ \quad Y(0) = 0. \end{cases} \tag{4}$$

To show that there is a unique solution to the IVP *Eq. (4)* where $0 \le X \le 2 + 2J$, holding X constant and applying the Mean Value Theorem to the function

$$\mathcal{F}(X, Y) = 1 + X\sin(XY).$$

We find that when $Z_1 < Z_2$, a number ξ in the interval (Z_1, Z_2) exists with

$$\frac{\mathcal{F}(X, Z_1) - \mathcal{F}(X, Z_2)}{Z_1 - Z_2} = \frac{\partial}{\partial Y}\mathcal{F}(X, \xi) = X^2\cos(\xi X).$$

Thus,

$$|F(X, Z_1) - \mathcal{F}(X, Z_2)| = |Z_1 - Z_2||X^2\cos(\xi X)| \le (2 + 2J)^2|Z_1 - Z_2|,$$

and $\mathcal{F}$ satisfies a Lipschitz condition in Y with a Lipschitz constant $L = (2 + 2J)^2$.

Moreover, $\mathcal{F}(X, Y)$ is continuous when $0 \le X \le 2 + 2J$ and $-\infty < Y_0, Y_1 < +\infty$, so *Theorem 2* implies that a unique solution exists to the IVP *Eq. (4)*.

Now, we can move to the next important considerations.

3.2 | Stability

In view of the stability analysis of the WFC-IVP *Eq. (1)*, by perturbing both the initial value C and the function $\mathcal{F}$, we consider the perturbed problem:

$$\begin{cases} \dfrac{dZ}{dX} = \mathcal{F}\big(X, Z(X)\big) + \delta(X), \\ Z(A) = C + \delta_C, \end{cases} \tag{5}$$

where $\delta(X)$ is a WFC continuous function for all $X \in [A, B]$ and the constant $\delta_C \in F_J$.

The sensitivity of the solution Z to the perturbations will be characterized in the following.

Definition 14. Let $I = [A, B] \subseteq F_J$ be a bounded set. The WFC Cauchy problem *Eq. (1)* is stable in I if, for any perturbation $(\delta_C, \delta(X))$ satisfying

$$|\delta_C| < E, \qquad |\delta(X)| < E, \qquad \forall X \in I.$$

With $E > 0$ suffciently small to guarantee that the solution to the perturbed problem *Eq. (5)* does exist, and $\exists k > 0$ constant depends in general on initial problem data A, C and $\mathcal{F}$ then

$$|Z(X) - Y(X)| < kE.$$

Lemma 2. The WFC-IVP *Eq. (1)* is a stable problem if its two corresponding IVPs *Eq. (2)* and *Eq. (3)* are stable.

Proof: the stability of *Eq. (2)* and *Eq. (3)* means that:

There exist constants $\varepsilon_0, \varepsilon_1 > 0$, and $k_0, k_1 > 0$ such that for any $\varepsilon, \varepsilon'$, with $\varepsilon_0 > \varepsilon > 0$ and $\varepsilon_1 > \varepsilon' > 0$, whenever $\delta_0(X_0), \delta_1(X_1)$ are continuous with $|\delta_0(X_0)| < \varepsilon$ for all $X_0 \in [A_0, B_0]$ and $|\delta_1(X_1)| < \varepsilon'$ for all $X_1 \in [A_1, B_1]$, and when $|\delta_{00}| < \varepsilon$ & $|\delta_{11}| < \varepsilon'$, the initial-value problems:

$$\begin{cases} \frac{dZ_0}{dX_0} = \mathcal{F}_1(X_0, Z_0) + \delta_0(X_0), \\ Z_0(A_0) = C_0 + \delta_{00}, \end{cases}$$

$$\begin{cases} \frac{dZ_1}{dX_1} = \mathcal{F}_2(X_1, Z_1) + \delta_1(X_1), \\ Z_1(A_1) = C_1 + \delta_{11}, \end{cases}$$

have unique solutions $Z_0(X_0)$ and $Z_1(X_1)$, respectively, that satisfy

$$|Z_0(X_0) - Y_0(X_0)| < k_0\varepsilon,$$

$$|Z_1(X_1) - Y_1(X_1)| < k_1\varepsilon'.$$

Using the isomorphic function φ:

there exist a WFC constant $E_c = \varphi^{-1}(\varepsilon_0, \varepsilon_1) > 0$ and $k = \varphi^{-1}(k_0, k_1) > 0$ such that for any $E = \varphi^{-1}(\varepsilon, \varepsilon')$, with $E_c > E > 0$, whenever $\delta(X) = \varphi^{-1}(\delta_0(X_0), \delta_1(X_1))$ is continuous with

$|\delta(X)| = \varphi^{-1}(|\delta_0(X_0)|, |\delta_1(X_1)|) < E$ for all $X \in [A, B]$, and when $|\delta_C| = \varphi^{-1}(|\delta_{00}|, |\delta_{11}|) < E$, the IVP (where $\delta_C = \varphi^{-1}(\delta_{00}, \delta_{11})$)

$$\begin{cases} \frac{dZ}{dX} = \mathcal{F}(X, Z) + \delta(X), \\ Z(A) = C + \delta_C, \end{cases}$$

has a unique solution $Z(X) = \varphi^{-1}(Z_0(X_0), Z_1(X_1))$, that satisfies

$$\varphi^{-1}(|Z_0(X_0) - Y_0(X_0)|, |Z_1(X_1) - Y_1(X_1)|) < \varphi^{-1}(k_0\varepsilon, k_1\varepsilon') \Rightarrow |Z(X) - Y(X)| < kE,$$

then, the WFC Cauchy problem *Eq. (1)* is stable.

3.3 | Well-Posedness

It is significant to know whether small perturbations, in the statement of the problem introduce parallelly small changes in the solution, that the effects of errors are bounded when the problem is well-posed.

Definition 15. The WFC-IVP

$$\begin{cases} dY = \mathcal{F}(X, Y)dX \\ Y(A) = C \end{cases} ; A \leq X \leq B$$

is said to be a well-posed problem if,

I. Its unique solution exists (existence and uniqueness).

II. Its solution depends continuously on the initial condition (stability).

Lemma 3. The WFC-IVP *Eq. (1)* is a well-posed problem if its two corresponding IVPs *Eq. (2)* and *Eq. (3)* are well-posed.

Proof: the well-posedness of *Eq. (2)* and *Eq. (3)* intends [9]:

I. The existence of the two unique solutions $Y_0(X_0)$ and $Y_1(X_1) \in R$ of *Eq. (2)* and *Eq. (3)*, respectively ⇒ the existence of the unique solution $Y(X) \in F_J$ of *Eq. (1)* (by *Lemma 1*).

II. The IVPs *Eq. (2)* and *Eq. (3)* are stable ⇒ the WFC-IVP *Eq. (1)* is stable (by *Lemma 2*).

III. which means that the WFC-IVP *Eq. (1)* is well-posed.

Theorem 3 (Well-Posed WFC-IVP). Suppose $D = \{(X, Y); X \in I;\ I = [A, B] \subseteq F_J, -\infty < Y_0, Y_1 < +\infty\}$. If $\mathcal{F}(X, Y)$ is continuous, and satisfies a Lipschitz condition on the set D in the variable Y, then the WFC-IVP *Eq. (1)* is a well-posed.

Proof: since $\mathcal{F}(X, Y)$ is continuous, and satisfies a Lipschitz condition on the set D, then

I. $\mathcal{F}_1, \mathcal{F}_2$ are continuous on $D_1, D_2 \subseteq R$.

II. $\mathcal{F}_1, \mathcal{F}_2$ satisfy Lipschitz conditions on D_1, D_2 in their own variables $Y_0(X_0), Y_1(X_1)$, respectively.

Then each IVP *Eq. (2)* and *Eq. (3)* is a well-posed (well-posed problem theorem [9]).

So, *Eq. (1)* is a WFC-IVP well-posed problem (*Lemma 3*).

Unless the original problem *Eq. (1)* is well-posed, there is a little reason to anticipate that the numerical solution to a perturbed problem *Eq. (5)* will provide accurate approximations to the original problem.

Since it is not always easy to find an explicit solution, we will use numerical methods which can be applied to any ODE with an initial condition to get a unique solution.

4 | Solving the First-Order Weak Fuzzy Complex-Initial Value Problems Numerically

We will combine the numerical solutions of the two IVPs *Eq. (2)* and *Eq. (3)* in R to get the numerical solution of WFC-IVP *Eq. (1)*

$$\begin{cases} dY = \mathcal{F}(X, Y)dX \\ Y(A) = C \end{cases} ; A \leq X \leq B.$$

4.1 | Numerical Methodology to Solve a Weak Fuzzy Complex-Initial Value Problem

The object of numerical methods is to obtain approximations $Z[i]$ and $S[i]$ to well-posed first-order IVPs *Eq. (2)* and *Eq. (3)*:

$$\begin{cases} dY_0 = \mathcal{F}_1(X_0, Y_0)dX_0, \\ Y_0(A_0) = C_0. \end{cases}$$

$$\begin{cases} dY_1 = \mathcal{F}_2(X_1, Y_1)dX_1, \\ Y_1(A_1) = C_1. \end{cases}$$

At $N + 1$ equally spaced numbers* in the intervals $[A_0, B_0], [A_1, B_1]$, respectively,

$$Z[i] \text{ and } S[i]; \qquad i = 0,1,2, \cdots, N.$$

At discrete sets of nodes,

$$X_0[0] = A_0 < X_0[1] < X_0[2] < \cdots < X_0[N] = B_0,$$
$$X_1[0] = A_1 < X_1[1] < X_1[2] < \cdots < X_1[N] = B_1.$$

*In our study, the two discrete sets have the same number of nodes. That, we will carefully choose the WFC numbers $A < B$ (borders of X) to give us the same value of step sizes $h_0 = h_1 = h$.

Taking these nodes in the form,

$$X_0[i] = A_0 + ih_0,\ X_1[i] = A_1 + ih_1;\ i = 0,1,2,\cdots,N,$$

where $h_0 = \frac{|B_0 - A_0|}{N}$ (the step size for the first IVP *Eq. (2)*) and $h_1 = \frac{|B_1 - A_1|}{N}$ (the second IVP *Eq. (3)*).

Using φ^{-1}, we get approximate solutions to the well-posed first-order WFC-IVP *Eq. (1)*, $W_i = \varphi^{-1}(Z[i], S[i])$ at a discrete set of nodes, $X[i] = \varphi^{-1}(X_0[i], X_1[i]);\ i = 0,1,2,\cdots,N$.

4.2 | Numerical Stability and Convergence

The reason that stability considerations are influential is that in each step after the first step of a numerical technique we begin again with a new IVP, where the initial condition is the approximate solution value computed in the previous step.

Definition 16. A method is stable when the results depend continuously on the initial data, i.e., for small changes in the initial conditions elicit only small changes in the solution; otherwise, it is unstable. Some methods are stable only for certain choices of initial data, and are called conditionally stable.

Definition 17. For a WFC-IVP *Eq. (1)*, we say that a numerical method is absolutely stable (zero-stable) if, for h fixed, the approximate solutions W_i remains bounded, i.e.,

$$|W_i| \to 0 \text{ when } X_i \to +\infty.$$

Lemma 4. If a numerical method is absolutely stable for each IVPs *Eq. (2)* and *Eq. (3)*, then it is an absolutely stable method for the WFC-IVP *Eq. (1)*.

Proof: the numerical method is absolutely stable for each IVP *Eq. (2)* and *Eq. (3)* means, for h_0 and h_1 fixed, each $Z[i]$ and $S[i]$ remains bounded [20], i.e.,

$$|Z[i]| \to 0 \text{ and } |S[i]| \to 0 \text{ when } X_0[i], X_1[i] \to +\infty$$

$$\Rightarrow |W_i| = \varphi^{-1}(|Z[i]|, |S[i]|) \to 0 \text{ when } X_i = \varphi^{-1}(X_{0i}, X_{1i}) \to +\infty.$$

Definition 18. A method is convergent in F_J if the approximation W_i approaches Y_i -the exact value of the solution of the differential equation at X_i as the step size goes to zero, i.e.,

$$\lim_{h \to 0} \max_{0 \le i \le N} |Y_i - W_i| = 0.$$

Lemma 5. If a method is convergent in R for each IVPs *Eq. (2)* and *Eq. (3)*, then it is convergent method for the WFC-IVP *Eq. (1)*.

Proof: we know that the exact value of the solution of a WFC-IVP is $Y[i] = \varphi^{-1}(Y_{0i}, Y_{1i})$ and the approximation is $W_i = \varphi^{-1}(Z[i], S[i])$ at X_i

When a method is convergent in R, the approximations $Z[i], S[i]$ approach Y_{0i}, Y_{1i} the solutions to IVPs *Eq. (2)* and *Eq. (3)*, respectively, as the steps size goes to zero ([9]). So that we can write

$$\varphi^{-1}\left(\lim_{h_0 \to 0} \max_{0 \le i \le N} |Y_{0i} - Z[i]|, \lim_{h_1 \to 0} \max_{0 \le i \le N} |Y_{1i} - S[i]|\right) = \varphi^{-1}(0,0)$$

$$\Rightarrow \lim_{h \to 0} \max_{0 \le i \le N} \varphi^{-1}(|Y_{0i} - Z[i]|, |Y_{1i} - S[i]|) = 0$$

$$\Rightarrow \lim_{h \to 0} \max_{0 \le i \le N} |Y_i - W_i| = 0$$

5 | Euler's Method to Solve the First-Order Weak Fuzzy Complex-Initial Value Problems

The most well-known and powerful numerical techniques used to solve IVPs are Taylor series expansions.

5.1 | Taylor Series Expansions Methods

Definition 19. Assuming that our solution WFC function $\mathrm{Y(X)}$ has $\mathrm{n+1}$ continuous derivatives, is represented by its Taylor series

$$\mathrm{Y(X+h) = Y(X) + hY'(X) + \frac{h^2}{2!}Y''(X) + \frac{h^3}{3!}Y'''(X) + \cdots + \frac{h^m}{m!}Y^{(m)}(X) + \cdots}.$$

We define the following concepts in $\mathrm{F_J}$,

I. when only terms through $\mathrm{\frac{h^m}{m!}Y^{(m)}(X)}$ are included in the Taylor series, the method that results is called the Taylor series method of order (m).

II. the difference equation can be expressed as

$$\mathrm{W_{i+1} = W_i + h\,F(X_i, W_i) + \frac{h^2}{2!}\mathcal{F}'(X_i, W_i) + \cdots + \frac{h^m}{m!}\mathcal{F}^{(m-1)}(X_i, W_i);\ i = 0,1,2,\cdots,N.}$$

III. the Taylor series method of order $\mathrm{m = 1}$ is known as Euler's method.

Errors will appear when applying numerical methods to find approximations [21].

Definition 20. At each step, if $\mathrm{W_i}$ is known, and $\mathrm{W_{i+1}}$ is computed from the first few terms of the Taylor series, an error occurs because we have truncated the Taylor series. This error, then, is called the local truncation error, and defined by

$$\mathrm{\tau = Y_i - W_i;\ i = 0,1,2,\cdots,N,}$$

where $\mathrm{Y_i = Y[i]}$ is the exact analytical solution in the step i, and $\mathrm{W_i}$ is the approximate solution in step i.

Definition 21. The absolute error is the absolute value of the difference between exact value and approximate value, i.e.,

$$\mathrm{e[i] = |Y_i - W_i|;\ i = 0,1,2,\cdots,N.}$$

5.2 | Euler's Algorithm in $\mathrm{F_J}$

In this paper, we will use Euler's method [18] to solve WFC-IVPs of the form *Eq. (1).*

Definition 22. Given the WFC-IVP of the form,

$$\begin{cases} \mathrm{dY = \mathcal{F}(X,Y)dX} \\ \mathrm{Y(A) = C} \end{cases} ;\mathrm{A \le X \le B},$$

where X -the WFC independent variable, we define Euler's algorithm to find $\mathrm{W_i}$ -the approximate solutions for this problem at the node points $\mathrm{X[i]}$, as following,

Input: endpoints $\mathrm{A, B}$; integer N; initial condition C.

Step 1. Set h -the step size, for each $\mathrm{i = 1,2,\cdots,N;\ N \in \mathbb{N}}$,

$$\mathrm{h = \frac{|B-A|}{N}},$$

$$\mathrm{X[0] = A},$$

$W_0 = C.$

Step 2. For $i = 1, 2, \ldots, N$ run *Steps 3* and *4*:

Step 3. Set $W_{i+1} = W_i + h\,\mathcal{F}(X[i], W_i)$. (Compute W_i) - the difference equation-

Step 4. $X[i] = A + ih$. (Compute $X[i]$)

Step 5. Output: approximations W_i to $Y[i]$ at the $N + 1$ values of $X[i]$.

Absolute Error: $e[i] = |Y_i - W_i|$

where the exact (analytical) solution of a WFC-IVP is $Y_i = Y[i]$.

Proof: we know that solving a WFC-IVP *Eq. (1)* is equivalent to solve the following IVPs in R

$$\begin{cases} dY_0 = \mathcal{F}_1(X_0, Y_0)dX_0, \\ Y_0(A_0) = C_0, \end{cases}$$

$$\begin{cases} dY_1 = \mathcal{F}_2(X_1, Y_1)dX_1, \\ Y_1(A_1) = C_1, \end{cases}$$

where

$$Y = \varphi^{-1}(Y_0, Y_1), \quad Y_0 = y_0 - y_1\sqrt{t},\ Y_1 = y_0 + y_1\sqrt{t},$$

$$X = \varphi^{-1}(X_0, X_1), \quad X_0 = x_0 - x_1\sqrt{t},\ X_1 = x_0 + x_1\sqrt{t},$$

$$A = \varphi^{-1}(A_0, A_1), \quad A_0 = a_0 - a_1\sqrt{t},\ A_1 = a_0 + a_1\sqrt{t},$$

$$B = \varphi^{-1}(B_0, B_1), \quad B_0 = b_0 - b_1\sqrt{t},\ B_1 = b_0 + b_1\sqrt{t},$$

$$C = \varphi^{-1}(C_0, C_1), \quad C_0 = c_0 - c_1\sqrt{t},\ C_1 = c_0 + c_1\sqrt{t}.$$

Then, Euler's Algorithm will be applied for each IVPs *Eq. (2)* and *Eq. (3)*, as the following:

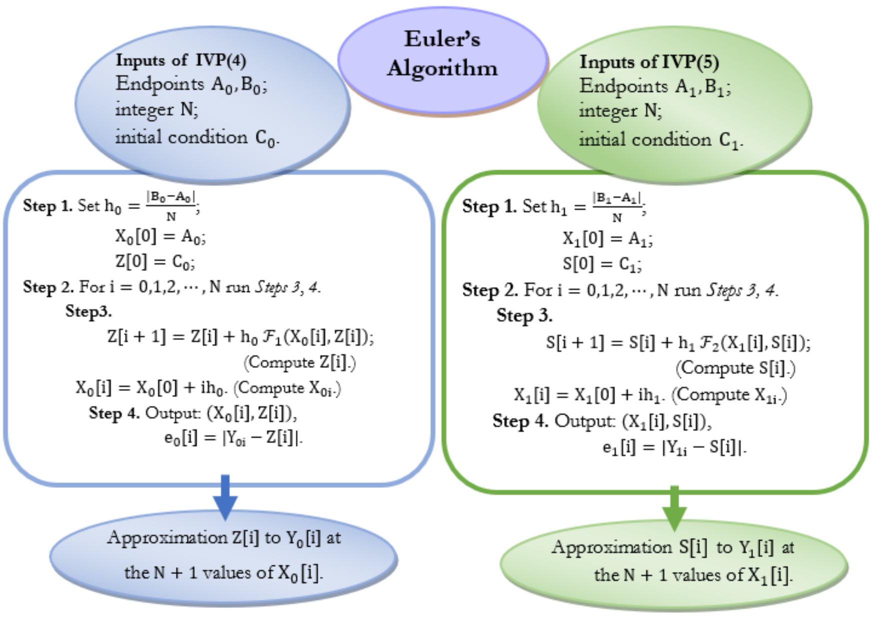

Fig. 1. Eluer's algorithm.

I. $h = \varphi^{-1}(h_0, h_1) = \frac{1}{N}\varphi^{-1}(|B_0 - A_0|, |B_1 - A_1|) = \frac{|B-A|}{N}$.

II. $X_i = X[i] = \varphi^{-1}(X_{0i}, X_{1i})$, where $X_{0i} = X_0[i]$, $X_{1i} = X_1[i]$,

$$X_i = \frac{1}{2}[X_{0i} + X_{1i}] + \frac{1}{2\sqrt{t}}J[X_{1i} - X_{0i}],$$
$$= \frac{1}{2}[X_0[0] + X_1[0] + 2ih] + \frac{1}{2\sqrt{t}}J[X_1[0] - X_0[0]],$$
$$= \frac{1}{2}\big[X_0[0] + X_1[0]\big] + \frac{1}{2\sqrt{t}}J\big[X_1[0] - X_0[0]\big] + ih,$$
$$= X[0] + ih; \quad X[0] = \varphi^{-1}(X_0[0], X_1[0]),$$
$$= A + ih.$$

III. Euler approximations $W_{i+1} = \varphi^{-1}(Z[i+1], S[i+1]); \quad i = 0,1,2,\cdots,N.$

By induction, we will prove that

$$W_{i+1} = W_i + h\,\mathcal{F}(X_i, W_i); \; i = 0,1,2,\cdots,N.$$

For $i = 0$ (describes the initial data), we have

$$W_0 = \varphi^{-1}(Z[0], S[0]) = \frac{1}{2}\big[Z[0] + S[0]\big] + \frac{1}{2\sqrt{t}}J\big[S[0] - Z[0]\big],$$
$$= \frac{1}{2}[C_0 + C_1] + \frac{1}{2\sqrt{t}}J[C_1 - C_0] = C.$$

Now, suppose it is true for i,

$$W_i = W_{i-1} + h\,\mathcal{F}(X_{i-1}, W_{i-1}),$$

and we will prove it for $i + 1$,

$$W_{i+1} = W_i + h\,\mathcal{F}(X_i, W_i).$$

We know that,

$$W_{i+1} = \varphi^{-1}(Z[i+1], S[i+1]),$$
$$= \frac{1}{2}[Z[i+1] + S[i+1]] + \frac{1}{2\sqrt{t}}J[S[i+1] - Z[i+1]],$$
$$= \frac{1}{2}\big[Z[i] + S[i]\big] + \frac{1}{2\sqrt{t}}J\big[S[i] - Z[i]\big] + \frac{1}{2}[h_0\,\mathcal{F}_1(X_0[i], Z[i]) + h_1\,\mathcal{F}_2(X_1[i], S[i])] +$$
$$\frac{1}{2\sqrt{t}}J[h_1\,\mathcal{F}_2(X_1[i], S[i]) - h_0\,\mathcal{F}_1(X_0[i], Z[i])],$$
$$= \varphi^{-1}(Z[i], S[i]) + \varphi^{-1}[(h_0, h_1)\big(\mathcal{F}_1(X_0[i], Z[i]), \mathcal{F}_2(X_1[i], S[i])\big)],$$
$$= W_i + h\,\mathcal{F}(X_i, W_i),$$

where

$$W_i = \varphi^{-1}(Z[i], S[i]), \; h = \varphi^{-1}(h_0, h_1), \; \mathcal{F}(X_i, W_i) = \varphi^{-1}\big(\mathcal{F}_1(X_0[i], Z[i]), \mathcal{F}_2(X_1[i], S[i])\big).$$

Abs-error $e[i] = \varphi^{-1}(e_0[i], e_1[i]) = \varphi^{-1}(|Y_{0i} - Z[i]|, |Y_{1i} - S[i]|) = |\varphi^{-1}(Y_{0i}, Y_{1i}) - \varphi^{-1}(Z[i], S[i])| = |Y_i - W_i|$

Table 1. Outputs of Euler's Algorithm for IVPs (the corresponding program using Python is in the Appendix).

i	X_{0i}	$Z[i]$	Y_{0i}	$e_0[i]$	X_{1i}	$S[i]$	Y_{1i}	$e_1[i]$
0	$X_{00} = 0.0$	1.0	1.0	0.0	$X_{10} = 0.0$	1.0	1.0	0.0
1	$X_{01} = 0.2$	0.8	0.8187307530	0.0187307530	$X_{11} = 0.2$	0.8	0.8187307530	0.0187307530
2	$X_{02} = 0.4$	0.64	0.6703200460	0.0303200460	$X_{12} = 0.4$	0.64	0.6703200460	0.0303200460
3	$X_{03} = 0.6$	0.512	0.5488116360	0.0368116360	$X_{13} = 0.6$	0.512	0.5488116360	0.0368116360
4	$X_{04} = 0.8$	0.4096	0.4493289641	0.0397289641	$X_{14} = 0.8$	0.4096	0.4493289641	0.0397289641
5	$X_{05} = 1.0$	0.32768	0.3678794411	0.0401994411	$X_{15} = 1.0$	0.32768	0.3678794411	0.0401994411
6	$X_{06} = 1.2$	0.2621440	0.3011942119	0.0390502119	$X_{16} = 1.2$	0.2621440	0.3011942119	0.0390502119
7	$X_{07} = 1.4$	0.20971520	0.2465969639	0.0368817639	$X_{17} = 1.4$	0.20971520	0.2465969639	0.0368817639
8	$X_{08} = 1.6$	0.167772160	0.2018965179	0.0341243579	$X_{18} = 1.6$	0.167772160	0.2018965179	0.0341243579
9	$X_{09} = 1.8$	0.1342177280	0.1652988882	0.0310811602	$X_{19} = 1.8$	0.1342177280	0.1652988882	0.0310811602
10	$X_{010} = 2.0$	0.1073741824	0.1353352832	0.02796110083	$X_{110} = 2.0$	0.10737418240	0.1353352832	0.02796110083

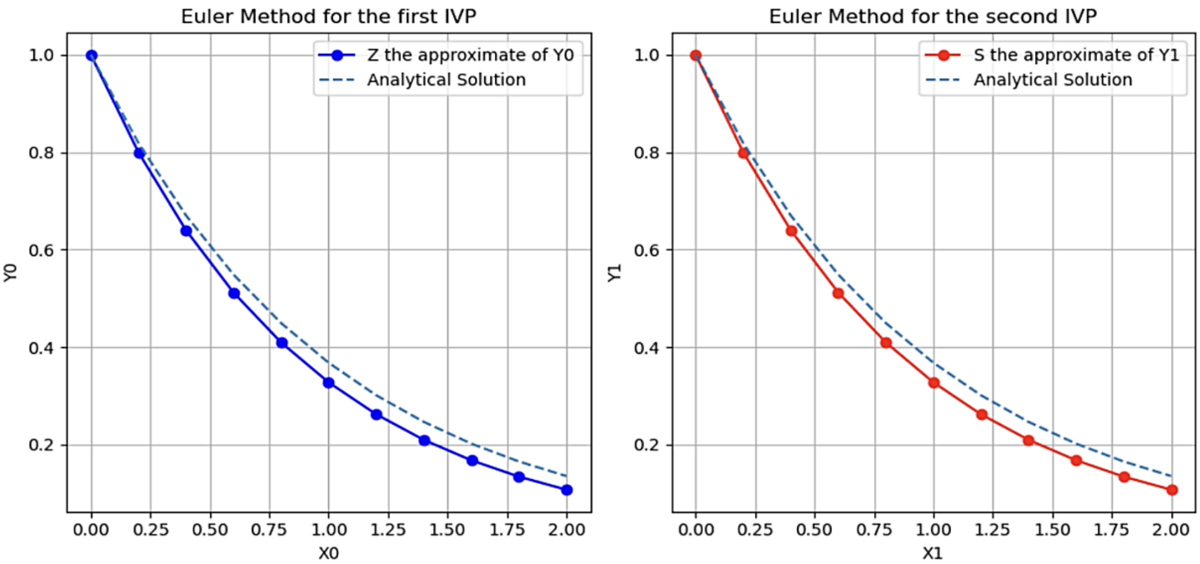

Fig. 2. The results of approximations, and exact solutions for IVPs.

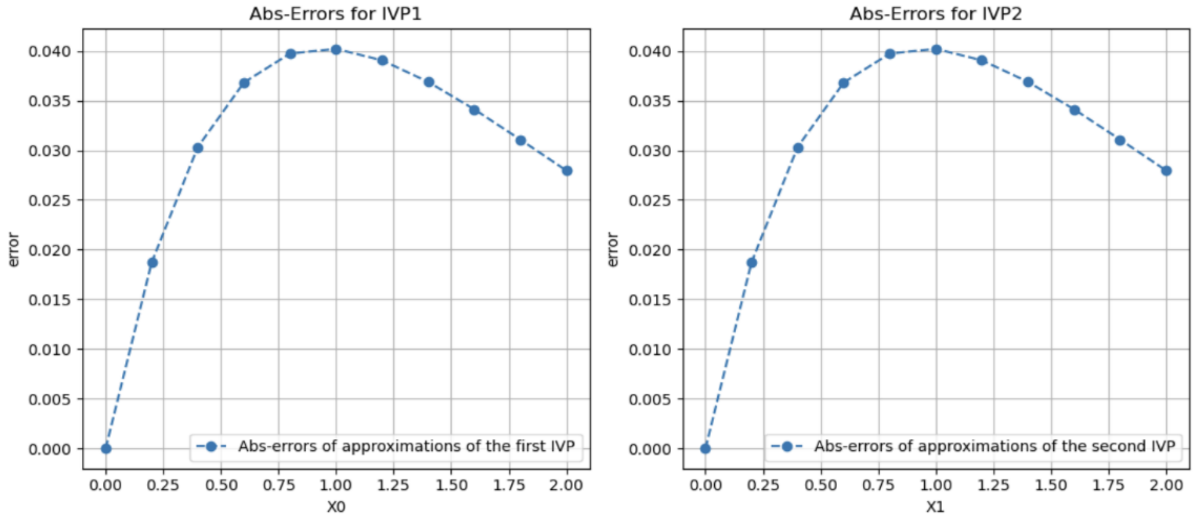

Fig. 3. The errors for IVPs.

Solving directly by Euler (the corresponding program is similar to previous programs).

Table 2. Outputs of Euler's Algorithm for IVP.

i	$\mathbf{X_i = \varphi^{-1}(X_{0_i}, X_{1_i})}$	$\mathbf{W_{i+1} = W_i + h\,\mathcal{F}(X_i, W_i) = \varphi^{-1}(Z[i], S[i])}$	$\mathbf{Y_i = e^{-X_i} = \varphi^{-1}(Y_{0i}, Y_{1i})}$	$\mathbf{Abs - error[i] = e[i] = \lvert Y_i - W_i \rvert = \varphi^{-1}(e_0[i], e_1[i])}$
0	$X_0 = 0.0$	1.0	1.0	0.0
1	$X_1 = 0.2$	0.8	0.8187307530	0.0187307530
2	$X_2 = 0.4$	0.64	0.6703200460	0.0303200460
3	$X_3 = 0.6$	0.512	0.5488116360	0.0368116360
4	$X_4 = 0.8$	0.4096	0.4493289641	0.0397289641
5	$X_5 = 1.0$	0.32768	0.3678794411	0.0401994411
6	$X_6 = 1.2$	0.2621440	0.3011942119	0.0390502119
7	$X_7 = 1.4$	0.20971520	0.2465969639	0.0368817639
8	$X_8 = 1.6$	0.167772160	0.2018965179	0.0341243579
9	$X_9 = 1.8$	0.1342177280	0.1652988882	0.0310811602
10	$X_{10} = 2.0$	0.10737418240	0.1353352832	0.02796110083

Note: Under the conditions of our WFC-IVP test (where $\lambda < 0$ and $\lambda, A, B, C \in R$), we get the same real results in tables for the three problems,

$$(X_i, W[i]) = (X_{0_i}, Z[i]) = (X_{1_i}, S[i]).$$

6| Conclusion

This paper included the main notions of the WFC-IVPs, and their related concepts of existence, uniqueness, stability, and well-posedness. We use a special isomorphic transformation function to construct the solution of a WFC-IVP from the solutions of the two related real IVPs analytically and numerically. Also, we introduced the definitions and lemmas of stable and convergent numerical methods to solve WFC-IVPs. Then, we used Euler's method to solve a simple stable linear model IVP, which has real coefficients, and real initial values. After that, a simple example explains how Euler's algorithm works to find approximations and absolute errors by Python.

However, we know that only two terms in the Taylor series are used in Euler's method which causes a large truncation error, and the results cannot be computed with much accuracy. Thus, we aim to use higher-order Taylor series methods in the future.

Consent for Publication

The author confirms consent for the publication of this work.

Ethics Approval and Consent to Participate

The author confirms that this research did not involve human participants or animal subjects.

Funding

This research has received no external funding.

Data Availability

The data used and analyzed during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The author declare no conflict of interest regarding the publications of this paper.

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Appendix

For results in *Table 1*: using Python

Instructions to solve the first IVP (Left\blue Part of *Table 1*)

```
#For the first IVP in R: ---------------------------(for outputs:(X0, Z))
import numpy as np
import matplotlib.pyplot as plt
#The corresponding function:
def F1(u, z):
    return -z
#The Parameters
u0 = 0       # Start time
z0 = 1       # Initial condition Y0[0]=z0
uN = 2       # End time
N = 10       # The number of iterations
print ('Euler Method Results')
#STEP1
h = float ((uN - u0) / N) #the step size
print ('h=', h)
print ('The initial values i=0  : (X0[0]=',u0,',Z[0]=',z0,')')
def euler_method1(F1 , u0, z0, N, uN):
    u = np.linspace(u0, uN, N + 1)
    z = np.zeros(N + 1)
    z[0] = z0
#STEP2
    for i in range (N):
        #STEP3&4
        z [i + 1] = z [i] + h * F1(u[i], z[i])
        print (' i=',i,',(X0[',i+1,']=',u[i+1],'Z[',i+1,']=', z[i+1],')')
    return u, z
# Solve using Euler's method
u, z = euler_method1(F1 , u0, z0, N, uN)
#EXACT solution
# Compute the analytical solution
def analytical_solution1(u):
    return np.exp(-u)
print ('The exact solutions')
for i in range(N+1):
    Y0 = analytical_solution1(u)
    print ('Y0[',i,']=',Y0[i])

# Calculate the error
error1 = np.abs(Y0 - z)
print ('Abs-Errors')
for i in range (N+1):
         print ('e[',i,']=', error1[i])
```

Instructions to solve the first IVP (Right\Green Part of *Table 1*)

```
#For the second IVP in R: ---------------------------(for outputs:(X1, S))
#The corresponding function:
def F2(v, s):
    return -s
#The Parameters
v0 = 0       # Start time
s0 = 1       # Initial condition Y1[0]=s0
vN = 2      # End time
N = 10       # The number of iterations
print ('Euler Method Results')
#STEP1
h = float ((vN - v0) / N) #the step size
print ('h=', h)
print ('The initial values i=0  : (X1[0]=',v0,',S[0]=',s0,')')
def euler_method2 (F2 , v0, s0, N, vN):
    v = np.linspace(v0, vN, N + 1)
    s = np.zeros(N + 1)
    s[0] = s0
#STEP2
    for i in range (N):
        #STEP3&4
        s [i + 1] = s [i] + h * F2(v[i], s[i])
        print (' i=',i,', (X1[',i+1,']=',v[i+1],'S[',i+1,']=',s[i+1],')')
    return v, s
# Solve using Euler's method
v, s = euler_method2 (F2 , v0, s0, N, vN)
#EXACT solution
# Compute the analytical solution
def analytical_solution2 (v):
    return np.exp(-v)
print ('The exact solutions')
for i in range(N+1):
    Y1 = analytical_solution2(v)
    print ('Y1[',i,']=',Y1[i])

# Calculate the error
error2 = np.abs(Y1 - s)
print ('Abs-Errors')
for i in range (N+1):
         print ('e[',i,']=', error2[i])
```