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The Computational Results of Fuzzy Subgroups of Nilpotent Finite (P-Groups) Involving Multiple Sums

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Abstract

The theory of fuzzy sets has a wide range of applications, one of which is that of fuzzy groups. Part of its applications is to provide formalized tools for dealing with the imprecision intrinsic to many problems. Denote the number of chains of subgroups of a finite group G which ends in G by h(G). The method of computing h(G) is based on the application of the Inclusion-Exclusion Principle. In this context , h(G) is actually referred to as the number of district fuzzy subgroups for the finite nilpotent p-group. This work is therefore designed as part to classify the nilpotent groups formed from the Cartesian products of p-groups through their computations. In this paper, the Cartesian products of p-groups were taken to obtain nilpotent groups. the explicit formulae is given for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group of order eight with a cyclic group of order of an n power of two for, which n is not less than three.

Keywords: Finite p-groups, Nilpotent group, Fuzzy subgroups, Dihedral group, Inclusion-exclusion principle, Maximal subgroups.

1|Introduction

1|Preliminaries

A group is nilpotent if it has a normal series of a finite length n.
That is, $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\}$, where $G_i/G_{i+1} \leq Z(G/G_{i+1})$.
By this notion, every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus,

(i) Any finite product of nilpotent group is nilpotent.

(ii) If G is nilpotent of a class c, then, every subgroup and quotient group of G is nilpotent and of class $\leq c$.

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Nilpotent groups are so called because the "adjoint action" of any element is nilpotent.

Proposition 1. (i) Every abelian group is nilpotent. (ii) Every finite p-group is nilpotent.

Lemma 1. Let G be a nilpotent group. Then every maximal subgroup of G is normal in G.

Definition 1. A maximal subgroup of a group G is a subgroup$M < G$ such that there is no subgroup H with $M < H < G$. Thus a maximal subgroup is a proper subgroup which is largest amongst the proper subgroups.

Thus a maximal subgroup is a proper subgroup which is largest amongst the proper subgroups. This tells us that the study of finite nilpotent groups reduces to understanding p-groups.
Theorem 1. Let G be a finite group. The following conditions on G are equivalent: (i) G is nilpotent; (ii) every Sylow subgroup of G is normal; (iii) G is a direct product of p-groups (for various primes p)

Proposition 2. Every nilpotent group is solvable

Theorem 2. Let G be a finite group. The following are equivalent: (i) G is nilpotent; (ii) every maximal subgroup of G is normal; (iii) G is a direct product of p -groups.
.

Theorem 3. (i) Every subgroup of a nilpotent group is nilpotent. (ii) Ever quotient group of a nilpotent group is nilpotent.

Proposition 3. If $G_1, \cdots, G_k$ are nilpotent groups then the direct product $G_1 \times \cdots \times G_k$ is also nilpotent.

Corollary 1. If $p_1, \cdots, p_k$ are primes and P_i is a p_i -group then $P_1 \times \cdots \times P_k$ is a nilpotent group.

Theorem 4. Let G be a finite group. The following conditions are equivalent. (i) G is nilpotent. (ii) Every Sylow subgroup of G is a normal subgroup. (iii) G isomorphic to the direct product of its Sylow subgroups.

Example 1. Finite p-groups are nilpotent.

1.1|Fuzzy Setsn

The notion of a fuzzy set is derived from the generalisation of the concept of a crisp set. Unlike in classical set theory where membership of an element of a set is viewed in binary terms of a bivalent nature (is a member of or is not a member of), the generalisation of classical sets to fuzzy sets allows for elements of a set to partially belong to that set.

Definition 2. Zadeh(1965): A fuzzy subset of a set X is a function $\mu : X \to I = [0, 1]$.
In an alternative manner, the fuzzy subset $\mu : X \to I = [0, 1]$ can be represented by $\mu_X(x) = t$ for $x \in X,\ 0 \leq t \leq 1$ and we say t is the degree to which x belongs to the fuzzy subset μ_X. This gives a crisp set if the image set is $\{0, 1\}$. We denote the set of all fuzzy sets of a set X by I^X.

Definition 3. Zadeh(1965): The Height of a fuzzy set
$ht(\mu) = \sup\{\mu(x) : x \in X\}$. We say the fuzzy set is normal if $ht(\mu) = 1$.

Definition 4. *(fuzzy set).* Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function

$$\mu_A : X \to [0, 1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.
It is clear that A is completely determined by the set of tuples

$A = \{(u, \mu_A(u)) | u \in X\}$. Set $A(x) = \mu_A(x)$. The family of all fuzzy sets in X is denoted by $F(X)$. If $X = \{x_1, \ldots, x_n\}$ is a finite set and A is a fuzzy set in X then the following notation is often used.

$$A = \mu_1/x_1 + \cdots + \mu_n/x_n,$$

where the term μ_i/x_i, $i = 1, \ldots, n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Example 2. *The membership function of the fuzzy set of real numbers "is close to 1", can be defined as*

$$A(t) = \exp(-\beta(t-1)^2)$$

where β is a positive real number.

Definition 5. (support) Let A be a fuzzy subset of X; the support of A, denoted $supp(A)$, is the crisp subset of X whose elements all have nonzero membership grades in A. $supp(A) = \{x \in X | A(x) > 0\}$.

Definition 6. (normal fuzzy set) A fuzzy subset A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.

Definition 7. (α-cut) An α-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]\alpha$ and is defined by

$$[A]\alpha = \begin{cases} \{t \in X | A(t) \geq \alpha\}, & \text{if } \alpha > 0, \\ cl(suppA), & \text{if } \alpha = 0, \end{cases}$$

where $cl(suppA)$ denotes the closure of the support of A.

Example 3. Assume $X = \{-2, -1, 0, 1, 2, 3, 4\}$ and $A = 0.0/-2 + 0.3/-1$

$$+0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4,$$

in this case

$$[A]\alpha = \begin{cases} \{-2, -1, 3, 4\}, & \text{if } 0 \leq \alpha \leq 0.3, \\ \{0, 2\}, & \text{if } 0.3 < \alpha \leq 0.6, \\ \{1\}, & \text{if } 0.6 < \alpha \leq 1. \end{cases}$$

Definition 8. (convex fuzzy set) A fuzzy set A of X is called convex if $[A]\alpha$ is a convex subset of $X \ \forall\, \alpha \in [0, 1]$. In many situations people are only able to characterize numeric information imprecisely.
For example, people use terms such as, about 5000, near zero, or essentially bigger than 5000. These are examples of what are called fuzzy numbers. Using the theory of fuzzy subsets we can represent these fuzzy numbers as fuzzy subsets of the set of real numbers.

Definition 9. (fuzzy number) A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by F.

Definition 10. (quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t \to \infty} A(t) = 0, \quad \lim_{t \to -\infty} A(t) = 0.$$

Let A be a fuzzy number. Then $[A]\gamma$ is a closed convex (compact) subset of R for all $\gamma \in [0, 1]$. Now, let

$$a_1(\gamma) = \min[A]\gamma, \quad a_2(\gamma) = \max[A]\gamma.$$

In other words, $a_1(\gamma)$ denotes the left-hand side and $a_2(\gamma)$ denotes the right-hand side of the γ-cut. It can be deduced that if $\alpha \leq \beta$ then, $[A]\alpha \supset [A]\beta$. Furthermore, the left-hand side function

$$\alpha_1 : [0, 1] \longrightarrow \mathrm{R}$$

is monotone increasing and lower semicontinuous, and the right-hand side function

$$\alpha_2 : [0,1] \longrightarrow \mathrm{R}$$

is monotone decreasing and upper semicontinuous.
Let

$$[A]\gamma = [a_1(\gamma),\ a_2(\gamma)].$$

The support of A is the open interval $(a_1(0), a_2(0))$.
If A is not a fuzzy number then there exists an $\gamma \in [0,1]$ such that $[A]\gamma$ is not a convex subset of R.

Definition 11. (triangular fuzzy number) A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha}, & \text{if } a - \alpha \le t \le a, \\ 1 - \frac{t-a}{\beta}, & \text{if } a \le t \le a + \beta, \\ 0, & \text{otherwise,} \end{cases}$$

and the notation $A = (a, \alpha, \beta)$ is thus, used. It can easily be verified that $[A]\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta]$, for all $\gamma \in [0,1]$. The support of A is $(a - \alpha, b + \beta)$. A triangular fuzzy number with center a may be seen as a fuzzy quantity " x is approximately equal to a".

Definition 12. (trapezoidal fuzzy number) A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form

$$A(t) = \begin{cases} 1 - (a-t)/\alpha, & \text{if } a - \alpha \le t \le a, \\ 1, & \text{if } a \le t \le b, \\ 1 - (t-b)/\beta, & \text{if } a \le t \le b + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and we use the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]\gamma = [\alpha - (1-\gamma)\alpha, b + (1-\gamma)\beta], \ \forall \ \gamma \in [0,1].$$

The support of A is $(a - \alpha, b + \beta)$. A trapezoidal fuzzy number may be seen as a fuzzy quantity
"x is approximately in the interval $[a, b]$".

Definition 13. Any fuzzy number $A \in F$ can be described as

$$A(t) = \begin{cases} L\left(\frac{a-t}{\alpha}\right)' & \text{if } t \in [a - \alpha, a], \\ 1, & \text{if } t \in [a, b], \\ R\left(\frac{t-b}{\beta}\right), & \text{if } t \in [b, b + \beta], \end{cases}$$

where $[a, b]$ is the peak or core of A,

$$L : [0,1] \longrightarrow [0,1], \ \ R : [0,1] \longrightarrow [0,1],$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. This fuzzy interval is called the LR-type . It is denoted by $A = (a, b, \alpha, \beta)_{LR}$. The support of A is $(a - \alpha, b + \beta)$.
Let $A = (a, b, \alpha, \beta)_{LR}$ be a fuzzy number of type LR. If $a = b$ then the notation $A = (a, \alpha, \beta)_{LR}$ is used in this case and hence, A is known as a quasi-triangular fuzzy number. Furthermore if $L(x) = R(x) = 1 - x$ then instead of $A = (a, b, \alpha, \beta)_{LR}$, simply write $A = (a, b, \alpha, \beta)$.

Definition 14. (subsethood) Let A and B be fuzzy subsets of a classical set X. We say that A is a subset of B if $A(t) \le B(t), \ \forall\, t \in X$.

Definition 15. (equality of fuzzy sets) Let A and B be fuzzy subsets of a classical set X. A and B are said to be equal, denoted $A = B$, if $A \subset B$ and $B \subset A$. We note that $A = B$ if and only if

$$A(x) = B(x) \text{ for } x \in X.$$

Definition 16. (empty fuzzy set) The empty fuzzy subset of X is defined as the fuzzy subset φ of X such that $\varphi(x) = 0$ for each $x \in X$. Observe that $\varphi \subset A$ holds for any fuzzy subset A of X.

Definition 17. The largest fuzzy set in X, called universal fuzzy set in X, denoted by 1_X, is defined by $1_X(t) = 1, \forall\, t \in X$. We have $A \subset 1_X$ holds for any fuzzy subset A of X.

Definition 18. (Fuzzy point) Let A be a fuzzy number. If $supp(A) = \{x_0\}$ then A is called a fuzzy point. The notation $A = \bar{x}_0$, is then used.

Let $A = \bar{x}_0$ be a fuzzy point. This implies that

$$[A]^\gamma = [x_0, x_0] = \{x_0\}, \ \ \forall\, \gamma \in [0, 1].$$

1.2|Operations on Fuzzy Sets

The classical set theoretic operations can be extended from ordinary set theory to fuzzy sets. All those operations which are extensions of crisp concepts are reduced to their usual meaning when the fuzzy subsets have membership degrees that are drawn from $\{0, 1\}$. For this reason, when extending operations to fuzzy sets the same symbols as in set theory are used. Let A and B be fuzzy subsets of a nonempty (crisp) set X.

Definition 19. (intersection) The intersection of A and B is defined as:
$(A \cap B)(t) = \min\{A(t), B(t)\} = A(t) \wedge B(t)$, for all $t \in X$.

Definition 20. (union) The union of A and B is defined as: $(A \cup B)(t) = \max\{A(t), B(t)\} = A(t) \vee B(t)$, for all $t \in X$.

Definition 21. (complement) The complement of a fuzzy set A is defined as

$$(\neg A)(t) = 1 - A(t)$$

A closely related pair of properties which hold in ordinary set theory are the law of excluded middle

$$A \vee \neg A = X$$

and the law of noncontradiction principle

$$A \wedge \neg A = \varphi$$

It is clear that $\neg 1_X = \varphi$ and $\neg \varphi = 1_X$, however, the laws of excluded middle and noncontradiction are not satisfied in fuzzy logic.

Lemma 2. The law of excluded middle is not valid. Let $pA(t) = \frac{1}{2}, \forall\, t \in R$,

$$\begin{aligned} \Rightarrow (\neg A \vee A)(t) &= \max\{\neg A(t), A(t)\} \\ &= \max\left\{1 - \frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2} \neq 1. \end{aligned}$$

Lemma 3. The law of noncontradiction is not valid.
Let $A(t) = 1/2, \ \ \forall\, t \in$ R, then this shows that

$$(\neg A \wedge A)(t) = \min\{\neg A(t), A(t)\} = min\{1 - \frac{1}{2}, \frac{1}{2}\} = 1/2 \neq 0.$$

However, fuzzy logic satisfies De Morgan's laws

$$\neg(A \wedge B) = \neg A \vee \neg B, \neg A \vee B = \neg A \wedge \neg B$$

1.3|Product Sets

Definition 22. An ordered pair can be intuitively defined as two elements x, y such that one of them is designated as the first element and the other as the second element. Such a pair is denoted by (x, y) with the understanding that two pairs (x, y) and (u, v) are equal if and only if $x = u$ and $y = v$.
In general, an ordered n-tuple written $(t_1, \ldots, t_n)$ can be considered as a set of n-elements $t_1, \ldots, t_n$ not

necessarily distinct, one of which is designated the first element, another the second element, etc. Also, if $(t_1, \ldots, t_n) = (t'_1, \ldots, t'_n)$ then it implies that $t_1 = t'_1, t_2 = t'_2, \ldots, t_n = t'_n$,

Definition 23. Suppose that P and Q are two sets. The Cartesian product (or product set) of P and Q, also written as $P \times Q$, is the set of all ordered pairs (x, y) such that $x \in P$ and $y \in Q$. Generally, the cartesian product $P_1 \times \cdots \times P_k$ of k sets $P_1, \ldots, P_k$ can be defined as the set of all n-tuples $(t_1, \ldots, t_k)$ for which, $t_i \in P_i$, $i \in \{1, 2, \ldots, k\}$. This means that $(t_1, \ldots, t_k) = (t'_1, \ldots, t'_k)$ if and only if $t_i = t'_i$ for all i.

Definition 24. Let ε be the set $\{1, 2, \ldots, n\}$ and $\{A_r\}_{r\in\varepsilon}$ a family of groups indexed by ε. The Cartesian product XA_r of $\{A_r\}$ is the set of all n-tuples $(x_1, \ldots, x_n)$, $x_i \in A_i$. This can alternatively be defined as the set of choice function from ε to $p\bigcup_{r=1}^{n} A_r$. Now, let $G = X_{r\in\epsilon}A_r$ be the Cartesian product of the groups A_r. We define a binary operation "$\star$" on G. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in G$, where $x * y = (x_1y_1, x_2y_2, \ldots, x_ny_n)$.

Theorem 5. [(Kuku(1992)] $(G, *)$ is a group. If each A_r is abelian, then G is abelian. If each A_r is finite of order b_t then G is finite of order $t_1t_2 \ldots t_n$.

Definition 25. The group $G = XA_r$ is known as the external direct product of the A's. This is denoted by $\Pi_{r\in\varepsilon}A_r$.

2|Introduction

The problem of classifying the fuzzy subgroups of a finite group has so far experienced a very rapid progress. One particular case or the other have been treated by several papers such as the finite abelian as well as the non-abelian groups. The number of distinct fuzzy subgroups of a finite cyclic group of square-free order has been determined. Moreover, a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups. They are the arbitrary finite cyclic groups and finite elementary abelian p-groups. For the first class, the explicit formula obtained gave rise to an expression of a well-known central Delannoy numbers. Some forms of propositions for classifying fuzzy subgroups for a class of finite p-groups have been made by Marius Tarnauceaus. It was from there, the study was extended to some important classes of finite non-abelian groups such as the dihedral and hamiltonian groups. And thus, a method of determining the number and nature of fuzzy subgroups was developed with respect to the equivalence relation. There are other different approaches for the classification. The corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups, and an essential role in solving counting problem is again played by the inclusion - exclusion principle. This hereby leads to some recurrence relations, whose solutions have been easily found. For the purpose of using the Inclusion - Exclusion principle for generating the number of fuzzy subgroups, the finite p-groups has to be explored up to the maximal subgroups. The responsibility of describing the fuzzy subgroup structure of the finite nilpotent groups is the desired objective of this work.

3|Methodology

We are going to adopt a method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G . That is the number of fuzzy subgroups of a finite group G which end in G. This is denoted by $h(G)$, and it is actually the number of district fuzzy subgroups for the finite nilpotent group .
Now , let G be a finite nilpotent group , and suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. In order to obtain $h(G)$, the simple application of the Inclusion-Exclusion Principle is thus put in place . and we have as follows :

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \tag{1}$$

In [6], (1) was used to obtain the explicit formulas for some positive integers n.
Theorem 6 (This can be credited to Marius . Please see [1]). The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:
(i) $h(\mathbb{Z}_{p^n}) = 2^n$, (ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = 2^{n-1}[2 + (n-1)p]$

We are going to apply this theorem at some points or the other in our computational processes .

4|The District Number of The Fuzzy Subgroups of The Nilpotent Group of $(D_{2^3} \times C_{2^m})$ For $m \geq 3$

The number of distinct fuzzy subgroups for the Cartesian product of the dihedral group of order eight and the abelian (cyclic) group of order 2^m for any integer $m \geq 3$ was fully computed in [14] . And so, the following propositions and theorem were used for the proof.

Proposition 4 (see [13]). Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \geq 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$

Proof: G has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$.
Hence, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$
$+ 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2})$

$$= \quad 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}[(n+1) - j]$$

= $2^{n+1}[2(n+1) + \frac{1}{2}(n-2)(n+3)] = 2^n[n^2 + 5n - 2], n \geq 2$

We have that : $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = 2^{n-1}[(n-1)^2 + 5(n-1) - 2]$
$= 2^{n-1}[n^2 + 3n - 6], n > 2$ □

Corollary 2. Following the last proposition, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8})$ = 1536, 4096, 10496 and 26112 respectively.

Theorem 7 (see [15]). Let $G = D_{2^n} \times C_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$

Proof: the group $D_{2^n} \times C_2$, has one maximal subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, two maximal subgroups which are isomorphic to $D_{2^{n-1}} \times C_2$, and 2^2 which are isomorphic to D_{2^n}.
It thus, follows from the Inclusion-Exclusion Principle using equation,

$\frac{1}{2}h(D_{2^n} \times C_2) = h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 4h(D_{2^n}) - 8h(D_{2^{n-1}}) - 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2h(D_{2^{n-1}} \times C_2)$

By recurrence relation principle we have :

$$h(D_{2^n} \times C_2) = 2^{2n}(2n+1) - 2^{n+1}, \quad n > 3$$

By the fundermental principle of mathematical induction,
set F(n) = $h(D_{2^n} \times C_2)$, assuming the truth of F(k) =$h(D_{2^k} \times C_2) = 2h(\mathbb{Z}_2 \times Z_{k-1})$
$+ 8h(D_{2^k} - 16hD_{2^{k-1}} - 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-2}) + 4h(D_{2^{k-1}} \times C_2) = 2^{2k}(2k+1) - 2^{k+1}$,
F(k+1) = $h(D_{2^{k+1}} \times C_2) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) + 8h(D_{2^{k+1}} - 16h(D_{2^k} - 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-1})$
$+ 4h(D_{2^k} \times C_2) = 2^2[2^{2k}(2k-3) - 2^k]$, which is true. □

Proposition 5 (see [12]). Suppose that $G = D_{2^n} \times C_4$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

Proof: $\frac{1}{2}h(D_{2^n} \times C_4) = h(D_{2^n} \times C_2) + 2h(D_{2^{n-1}} \times C_4) - 4h(D_{2^{n-1}} \times C_2) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$- 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 8h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + h(\mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^{n-2}})$

$$h(D_{2^n}\times C_4)=(n-3).2^{2n+2}+2^{2(n-3)}(1460)+3[2^n(2n-1)+2^{n+1}(2n-3)+2^{n+2}(2n-5)+\cdots+7(2^{2(n-2)})]$$

$$=(n-3).2^{2n+2}+2^{2(n-3)}(1460)+3\sum_{j=1}^{n-3}2^{n-1+j}(2n+1-2j)$$

$$=2^{2(n-2)}(64n+173)+3\sum_{j=1}^{n-3}2^{n-1+j}(2n+1-2j)$$

Proposition 6 (see [10]). Let G be an abelian p-group of type $\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^n}$, where p is a prime and $n\geq 1$. The number of distinct fuzzy subgroups of G is
$h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^n})=2^np(p+1)(n-1)(3+np+2p)+(2^n-2)p^3-2^{n+1}(n-1)p^3+2^n[p^3+4(1+p+p^2)].$

Proof: there exist exactly $1+p+p^2$ maximal subgroups for the abelian type $\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^n}$, [Berkovich(2008)]. One of them is isomorphic to
$\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-1}}$, while each of the remaining $p+p^2$ is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_{p^n}$. Thus, by the application of the Inclusion-Exclusion Principle,we have as follows: $h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^n})=2^np(p+1)(n-1)(3+np+2p)+(2^n-2)p^3-2^{n+1}(n-1)p^3+2^n[p^3+4(1+p+p^2)]$ And thus,

$$h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}})\;=\;2^{n-2}[4+(3n-5)p+(n^2-5)p^2+(n^2-5n+8)p^3]-2p^2.$$

□

Corollary 3. From (3) above, obsreve that, we are going to have that:

$$h(\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{3^n})\;=\;2^{n+1}[18n^2+9n+26]-54$$

Similarly, for $p=5$, using the same analogy, we have

$$\begin{aligned}h(\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_{5^n})\;&=\;2[30h(\mathbb{Z}_5\times\mathbb{Z}_{5^n})+h(\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_{5^{n-1}})\\&\quad-p^3h(\mathbb{Z}_{5^n})-30h(\mathbb{Z}_{5^{n-1}})+125]\end{aligned}$$

And for $p=7$,

$$h(\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_{7^n})=2[56h(\mathbb{Z}_7\times\mathbb{Z}_{7^n})+h(\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_{7^{n-1}})-343h(\mathbb{Z}_{7^n})-56h(\mathbb{Z}_{7^{n-1}})+343]$$

We have, in general, $h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_{p^{n-2}})=2^{n-2}[4+(3n-5)p+(n^2-5)p^2+(n^2-5n+8)p^3]-2p^2$ □

Proposition 7 (see [14]).
Let $G=(D_{2^3}\times C_{2^m})$ for $m\geq 3$. Then , $h(G)=m(89-23m)+(85)2^{m+3}-124$

Proof: there exist seven maximal subgroups , of which one is isomorphic to $D_{2^3}\times C_{2^m-1}$, two being isomorphic to $C_{2^m}\times C_2\times C_2)$, two isomorphic to $C_{2^m}\times C_2$, and one each isomorphic to $C_{2^m}\times C_4$, and C_{2^m} respectively. Hence , by the inclusion - exclusion principle, using the propositions [1], [2], [3], and Theorem 2 , we have that : $h(D_{2^3}\times C_{2^m})=(46m-3).2^{m+1}+2^6+(46m-49)2^{m+1}+2^7+(46m-95)2^{m+1}+2^8+2^3h(D_{2^3}\times C_{2^{m-3}})$
$=2^{m+1}.[(46m-3)+(46m-49)+(46m-95)]+2^6+2^7+2^8+2^3h(D_{2^3}\times C_2^{m-3})$

$$\begin{aligned}=&\quad\underbrace{2^6+2^7+2^8+\cdots+2^{5+k}}_{\text{series (1)}}\\&+\;2^{m+1}.[46mk+\underbrace{(-3-49-95\cdots(-3-46(k-1)))]}_{\text{series (2)}}\\&+2^kh(D_{2^3}\times C_{2^m-k}),k\in\{1,2.3.\cdots n\in\;N\}\end{aligned}$$

For the series (1) , we have that, $U_m=2^6.2^{m-1}=2^{5+k},m+5=k+5,\Rightarrow m=k$. We have that $S_{m=k}=2^6[\frac{2^k-1}{2-1}]=2^6(2^k.1)$

And for the series (2), we have that , $T_m=-3+(m-1)(-46)=-3-46(k-1)\Rightarrow m-1=k-1,n=k$ Hence , $S_m=k=\frac{k}{2}[2(-3)+(k-1)(-46)]=\frac{k}{2}(-6-46k+46)=\frac{k}{2}(40-46k)$, We have that

$h(D_{2^3}\times C_{2^m}) = \frac{k}{2}(40-46k)+2^6(2^k.1)+2^k h(D_3\times C_{2^m-k}$. By setting $m = k$ we have that $k = m-3$.Hence , $h(D_{2^3} \times C_{2^m}) = (m-3)(20-23m) + 2^6(2^{m-3}-1) + 2^m - 3h(D_3 \times C_{2^3})$

$h(G) = (m-3)(20-23m)+2^6(2^{m-3}-1)+2^{m-3}(5376) = (m-3)(20-23m)+2^{m-3}-2^6+2^{m+5}(21)$ = $20m-23m^2-60+69m+2^{m+3}-2^6+(21)2^{m+5}$ = $(89m-23m^2-60)+2^{m+3}-2^6+(21)2^{m+5}$ = $m(89-23m)-124+(85)2^{m+3}$ □

Theorem 8 (see [11]). Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3+12n^2+17n-24)$
Proof: the three maximal subgroups of G have the following properties :
one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})$, while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n})$.
We have : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
Hence , $h(G) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}})$
$+ 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}})$, for $n-j=3$

$$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})$$

$= 2^{n+2}[n^2+5n+3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}((n^2+5n+3) + \frac{1}{6}(n-3)(n^2+9n+14))$
$= \frac{1}{3}(2^{n+1})(n^3+12n^2+17n-24), n > 2.$ □

Proposition 8 (see [16]). Suppose that $G = D_{2^n} \times C_8$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-1)}(6n+113) + 2^n[13-6n-2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)]$$

$$+\frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

Proof. $h(D_{2^n} \times C_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8)$
$+ 4h(Z_{2^{n-2}} \times C_8) + 2^4h(Z_{2^{n-3}} \times C_8) + 2^6h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3})$
$- 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9h(\mathbb{Z}_{2^{n-5}}) - 2^9h(D_{2^{n-4}} \times C_{2^2})$
$+ 2^8h(D_{2^{n-4}} \times C_{2^3})$
$= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3})$
$- 2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2})$
$- 2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3})$

$$+ \quad 3\sum_{i=1}^{n-5} 2^{2ij}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})$$

as required. □

Theorem 9. Let $G = D_{2^4} \times C_{2^4}$. Then , $h(G) = 61384$
Proof: there exist seven maximal subgroups . Two isomorphic to $D_{2^4} \times C_{2^3}$. two isomorphic to $D_{2^3} \times C_{2^4}$. two isomorphic to $D_{2^4} \times C_{2^2}$, while the seventh is isomorphic to $\mathbb{Z}_{2^4}$. Hence , we have that : $\frac{1}{2}h(G) = 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times Z_{2^4}) - 6h(D_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4}) + 2h(D_{2^3} \times \mathbb{Z}_{2^3}) + 28h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_{2^4}) + h(Z_{2^3} \times \mathbb{Z}_{2^3}) - 35h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^3}\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2})$
= $2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 4h(D_{2^3} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^4}) + \frac{1}{2}h(Z_{2^4})]$

$\therefore h(G) = 4[700 + 8416 + 10744 - 10752˘1088 + 162 + 704˘40]$
$= 4[15346] = 61384$ □

5|Computation for $G = D_{2^4} \times C_{2^n}, n \geq 4.$

Our computation on the algebraic fuzzy structure given actually has an outcome which involves multiple sums. As usual, there exist seven maximal subgroups of which their intersections were constructed using GAP(Group Algorithms and Programming). This was then followed by applying the Inclusion-Exclusion principle.
We have as follows :
The maximal subgroups are : $(D_{2^4} \times C_{2^{n-1}}), 2(D_{2^3} \times C_{2^n}), 2(D_{2^n} \times C_{2^2}), (D_{2^n} \times C_{2^3})$ and (C_{2^n}). We have that : $\frac{1}{2}h(G) = h(D_{2^4} \times C_{n-1}) + 2h(D_{2^3} \times C_n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) + h(C_{2^n}) - 6h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 6h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^{n˘1}} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^n}) + 2h(D_{2^3} \times C_{2^{n-1}}) + 28h(C_{2^{n-1}} \times C_{2^n}) + h(C_{2^{n-1}} \times C_{2^3}) + 2h(C_{2^n} \times C_{2^2}) + 2h(\mathbb{Z}_{2^n}) - 35h(C_{2^{n-1}} \times C_{2^2}) + 21h(C_{2^{n-1}} \times C_{2^2}) - 7h(C_{2^{n-1}} \times C_{2^2}) + h(C_{2^{n-1}} \times C_{2^2})$

$= h(D_{2^4} \times C_{2^{n-1}}) + 2h(D_{2^3} \times C_2^n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) - 4h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^{n˘1}} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^n})$

$$\frac{1}{2}h(G) = h(D_{2^4} \times \mathbb{Z}_{2^{n-k}}) + 2h(D_{2^3} \times \mathbb{Z}_{2^n}) - 4h(D_{2^3} \times \mathbb{Z}_{2^{n-k}}) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2})$$

$$-2h(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^2}) + \sum_{j=1}^{k} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 2\sum_{j=1}^{k} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 3\sum_{j=1}^{k} h(\mathbb{Z}_{2^{n+1-j}})$$

$$-2\sum_{j=1}^{k-1} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 4\sum_{j=1}^{k-1} h(D_{2^{n-j}} \times \mathbb{Z}_{2^2}) - 2\sum_{j=1}^{k-1} h(D_{2^n-j} \times \mathbb{Z}_{2^3}),$$

whence , $n - k = 4, \Rightarrow k = n - 4. \therefore h(G) = 2h(D_{2^4} \times \mathbb{Z}_{2^4}) + 4h(D_{2^3} \times \mathbb{Z}_{2^n}) - 8h(D_{2^3} \times \mathbb{Z}_{2^4}) - 8h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 16h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) +$

$$2\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 4\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6\sum_{j=1}^{n-4} h(\mathbb{Z}_{2^{n+1-j}})$$

$$-4\sum_{j=1}^{n-5} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 8\sum_{j=1}^{n-5} h(D_{2^n-j} \times \mathbb{Z}_{2^2}) - 4\sum_{j=1}^{n-5} h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})$$

$$\therefore h(G) = 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + 2\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3})$$

$$+4\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6\sum_{j=1}^{n-4} h(\mathbb{Z}_{2^{n+1-j}}) - 4\sum_{j=1}^{n-5} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 8\sum_{j=1}^{n-5} h(D_{2^n-j} \times \mathbb{Z}_{2^2}) - 4\sum_{j=1}^{n-5} h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})$$

$$= 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + \sum_{j=1}^{n-4}[2h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 4h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6h(\mathbb{Z}_{2^{n+1-j}})]$$

$$-\sum_{j=1}^{n-5}[4h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) - 8h(D_{2^n-j} \times \mathbb{Z}_{2^2}) + 4h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})]$$

Hence , proved as required □

6|Applications

The computations so far by the use of GAP (General AlgorithmAlgorithms and Programming) and the Inclusion - Exclusion Principle can be certified here as being very useful in the computations of the district number of fuzzy subgroups for the finite nilpotent p - groups .

7|Instances

We have the following examples as parts surfacing from our computations so far. The readers may consider the examples below in tabular format.

Example 4. Now, since the stipulated condition that $m \geq 3$ must definitely be fulfilled then the readers may consider the examples below in tabular format.

Table 1
Table Summarizing some Number of Distinct Fuzzy Subgroups of $(D_{2^4} \times C_{2^n})$ **FOR** $n \geq 4$

S/N for the Number of n	4	5	6
$h(G) = (D_{2^4} \times C_{2^n}), n \geq 4$	20, 200	375, 648	3, 893, 800

8|Conclusion

The discoveries from our studies so far , has helped to observe that any finite product of nilpotent group is nilpotent. Also, the problem of classifying the fuzzy subgroups of a finite group has experienced a very rapid progress. Finally, the method can be used in further computations up to the generalizations of similar and other given structures.

Authors' Contributions

S. A. A.: Writing-original draft, Conceptualization, Data Curation, and Validation. M. O.: Research Design, Computing, and Editing. M. E.: Methodology, Visualization and Formal Analysis. The authors have read and agreed to the published version of the manuscript.

Consent for Publication

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Conflict of Interest

The authors declare that in this paper, there is no competing of interests.

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