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HyperRough TOPSIS Method and SuperHyperRough TOPSIS Method

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Abstract

Rough set theory provides a mathematical framework for approximating subsets through lower and upper bounds defined by equivalence relations, effectively capturing uncertainty in classification and data analysis. Building on these foundational ideas, extended models such as Hyperrough Sets and Superhyperrough Sets have been proposed to represent more complex forms of uncertainty.

In this paper, we introduce the HyperRough TOPSIS Method and the SuperHyperRough TOPSIS Method, and examine their underlying mathematical structures. TOPSIS is a well-established method in decision-making, and the proposed HyperRough and SuperHyperRough TOPSIS methods serve as generalized extensions of the classical Rough TOPSIS approach.

Keywords: Rough set, Hyperrough set, Rough TOPSIS, SuperHyperRough set, HyperRough TOPSIS, SuperHyperRough TOPSIS.

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1| Introduction

1.1| Rough Set Theory

Numerous set-theoretic frameworks have been developed to handle uncertainty. In this paper, we focus on Rough Set theory. A rough set approximates a target subset by computing lower and upper bounds from equivalence classes, thereby capturing both certain and uncertain membership [16]. Related approaches include Fuzzy Sets [28], Intuitionistic Fuzzy Sets [3], Neutrosophic Sets [21, 22], Plithogenic Sets [23, 24], Graded Rough Sets [7, 27], Multigranular Rough Sets [18], and Soft Sets [15]. These models have been extensively studied in various applications, including decision making and artificial intelligence.

Extensions of Rough Sets—such as the HyperRough Set and the n-SuperHyperRough Set—have also been proposed. The *HyperRough Set* extends classical rough set theory by incorporating multiple attribute domains. An n-SuperHyperRough Set further generalizes the concept by using iterative power sets of attribute values to produce more nuanced approximations under uncertainty [11].

1.2| Our Contribution

In this paper, we introduce the HyperRough TOPSIS method and the SuperHyperRough TOPSIS method, and we examine their underlying mathematical structures. TOPSIS is a well-established decision-making technique, and our proposed methods constitute generalized extensions of the classical Rough TOPSIS approach. We hope that these contributions will help advance research in decision-making theory.

2| Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets under consideration are assumed to be finite.

2.1| Rough Set, HyperRough Set, and SuperHyperRough Set

The definitions of the Rough Set, HyperRough Set, and SuperHyperRough Set are presented below.

Definition 2.1 (Universal Set)**.** A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

Definition 2.2 (Rough Set Approximation)**.** [17] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

(1) *Lower Approximation*:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

(2) *Upper Approximation*:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [11, 9].

Definition 2.3 (HyperRough Set)**.** [11] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 2.4 (HyperRough Set in Patient Diagnosis)**.** (cf.[2, 1]) Let

$$X = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$$

be eight patients in a hospital, and consider two attributes:

$$T_1 = \text{“Symptom Group”}, \quad J_1 = \{\text{Respiratory, Gastrointestinal}\},$$

$$T_2 = \text{“Test Result”}, \quad J_2 = \{\text{Positive, Negative}\}.$$

Define

$$J = J_1 \times J_2 = \{\, (s, t) \mid s \in J_1,\ t \in J_2 \}\,,$$

and let the equivalence relation R group patients by their ward:

$$[P_1]_R = \{P_1, P_2\},\ [P_3]_R = \{P_3, P_4\},\ [P_5]_R = \{P_5, P_6, P_7\},\ [P_8]_R = \{P_8\}.$$

Define $F : J \to \mathcal{P}(X)$ by

$$F(\text{Respiratory, Positive}) = \{P_1, P_3, P_5\},$$
$$F(\text{Respiratory, Negative}) = \{P_2, P_4\},$$
$$F(\text{Gastrointestinal, Positive}) = \{P_6, P_7\},$$
$$F(\text{Gastrointestinal, Negative}) = \{P_8\}.$$

Then for each $a \in J$ we compute

$$\underline{F(a)} = \{\, x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{\, x \in X \mid [x]_R \cap F(a) \neq \varnothing\}.$$

Concretely:

$$\underline{F}(\text{Resp+}) = \varnothing, \quad \overline{F}(\text{Resp+}) = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\},$$
$$\underline{F}(\text{Resp–}) = \varnothing, \quad \overline{F}(\text{Resp–}) = \{P_1, P_2, P_3, P_4\},$$
$$\underline{F}(\text{Gastro+}) = \varnothing, \quad \overline{F}(\text{Gastro+}) = \{P_5, P_6, P_7\},$$
$$\underline{F}(\text{Gastro–}) = \varnothing, \quad \overline{F}(\text{Gastro–}) = \{P_8\}.$$

One checks:

$$\underline{F(a)} \subseteq \overline{F(a)}, \quad \underline{F(a)} = \overline{F(a)} = \varnothing \text{ if } F(a) = \varnothing, \quad \underline{F(a)} = \overline{F(a)} = X \text{ if } F(a) = X.$$

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty[10, 8, 11]. The definition of n-SuperHyperRough Sets is described as follows.

Definition 2.5 (n-SuperHyperRough Set)**.** [11] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 2.6 (n-SuperHyperRough Set in Smartphone Selection)**.** (cf.[14, 4]) Let

$$X = \{S_1, S_2, S_3, S_4, S_5, S_6\}$$

be six smartphone models. Two attributes:

$$T_1 = \text{“Brand”}, \quad J_1 = \{\text{A}, \text{B}\},$$
$$T_2 = \text{“Screen Size”}, \quad J_2 = \{\text{Small}, \text{Medium}, \text{Large}\}.$$

We form

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2),$$

so each element $A = (A_1, A_2)$ has $A_1 \subseteq J_1$, $A_2 \subseteq J_2$. Let the equivalence relation R group models by release-year cohort:

$$[S_1]_R = \{S_1, S_3\}, \ [S_2]_R = \{S_2, S_6\}, \ [S_4]_R = \{S_4, S_5\}.$$

Define $F : J \to \mathcal{P}(X)$ on four representative combinations:

$$F(\{A\}, \{M, L\}) = \{S_1, S_2\},$$
$$F(\{B\}, \{S, M\}) = \{S_3, S_4, S_5\},$$
$$F(\{A, B\}, \{L\}) = \{S_2, S_4, S_6\},$$
$$F(\varnothing, \varnothing) = \varnothing.$$

Then for each $A \in J$,

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \varnothing\}.$$

For example:

$$\underline{F}(\{A\}, \{M, L\}) = \varnothing, \quad \overline{F}(\{A\}, \{M, L\}) = \{S_1, S_2, S_3, S_6\},$$
$$\underline{F}(\{B\}, \{S, M\}) = \{S_4, S_5\}, \quad \overline{F}(\{B\}, \{S, M\}) = \{S_1, S_3, S_4, S_5\},$$
$$\underline{F}(\{A, B\}, \{L\}) = \{S_2, S_4\}, \quad \overline{F}(\{A, B\}, \{L\}) = \{S_2, S_4, S_6\},$$
$$\underline{F}(\varnothing, \varnothing) = \overline{F}(\varnothing, \varnothing) = \varnothing.$$

One verifies the general properties:

$$\underline{F(A)} \subseteq \overline{F(A)}, \quad \underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

2.2| Rough TOPSIS Method

Research on decision-making and multi-criteria decision-making (MCDM) has gained significant attention in recent years[12, 13, 6]. Among various approaches, this paper focuses on the TOPSIS method. The Rough TOPSIS Method is a multi-criteria decision-making technique that employs interval-based approximations derived from rough set theory to effectively represent and manage uncertainty [26, 5, 20, 19]. The following section presents the formal definition of the method, along with a concrete illustrative example.

Definition 2.7 (Rough TOPSIS Method)**.** [26] Let $A = \{A_1, \ldots, A_m\}$ be a set of alternatives and $C = \{c_1, \ldots, c_n\}$ a set of criteria. Each performance rating is given as a rough number

$$\tilde{x}_{ij} = \left[\underline{x}_{ij}, \overline{x}_{ij}\right], \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

The Rough TOPSIS method proceeds as follows:

(1) **Construct decision matrix.**

$$\tilde{X} = (\tilde{x}_{ij})_{m \times n}.$$

(2) **Normalize.** For benefit criteria,

$$\tilde{r}_{ij} = \Big[\frac{\underline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \overline{x}_{ij}^2}}, \frac{\overline{x}_{ij}}{\sqrt{\sum_{i=1}^{m} \overline{x}_{ij}^2}}\Big].$$

(For cost criteria one takes reciprocals before normalization.)

(3) **Determine weights.** Let $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ be a weight vector of rough numbers $\tilde{w}_j = [\underline{w}_j, \overline{w}_j]$.

(4) **Weighted normalized matrix.** Compute

$$\tilde{v}_{ij} = \tilde{w}_j \ \times \ \tilde{r}_{ij}, \quad \text{via interval multiplication.}$$

(5) **Ideal solutions.**

$$\tilde{v}_j^+ = \left[\max_i \underline{v}_{ij}, \max_i \overline{v}_{ij}\right], \quad \tilde{v}_j^- = \left[\min_i \underline{v}_{ij}, \min_i \overline{v}_{ij}\right].$$

(6) **Distance measure.** For two rough numbers $\tilde{a} = [\underline{a}, \overline{a}]$, $\tilde{b} = [\underline{b}, \overline{b}]$, define

$$d(\tilde{a}, \tilde{b}) = \sqrt{(\underline{a} - \underline{b})^2 + (\overline{a} - \overline{b})^2}.$$

(7) **Distance to ideals.**

$$D_i^+ = \sqrt{\sum_{j=1}^{n} \left[d(\tilde{v}_{ij}, \tilde{v}_j^+)\right]^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{n} \left[d(\tilde{v}_{ij}, \tilde{v}_j^-)\right]^2}.$$

(8) **Closeness coefficient.**

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}, \quad \text{and rank alternatives in descending order of } CC_i.$$

Example 2.8. Consider three alternatives A_1, A_2, A_3 evaluated on two benefit criteria c_1, c_2. The normalized rough decision matrix and crisp weight vector $w = (0.4, 0.6)$ are

$$\tilde{r}_{ij} = \begin{pmatrix} [0.50, 0.70] & [0.60, 0.80] \\ [0.60, 0.80] & [0.50, 0.70] \\ [0.40, 0.60] & [0.70, 0.90] \end{pmatrix}, \quad w = (0.4, 0.6).$$

Compute the weighted normalized matrix

$$\tilde{v}_{ij} = w_j \cdot \tilde{r}_{ij} = \begin{pmatrix} [0.20, 0.28] & [0.36, 0.48] \\ [0.24, 0.32] & [0.30, 0.42] \\ [0.16, 0.24] & [0.42, 0.54] \end{pmatrix}.$$

The RPIS and RNIS are

$$\tilde{v}^{+} = \big([0.24, 0.32], [0.42, 0.54]\big), \quad \tilde{v}^{-} = \big([0.16, 0.24], [0.30, 0.42]\big).$$

Calculating distances and closeness coefficients yields

$$CC_1 = 0.50, \quad CC_2 = 0.40, \quad CC_3 = 0.60,$$

so the final ranking is

$$A_3 \succ A_1 \succ A_2.$$

3| Results of This Paper: HyperRough TOPSIS Method and SuperHyperRough TOPSIS Method

This section presents the results obtained in this paper.

3.1| HyperRough TOPSIS method

The definition of the HyperRough TOPSIS method is presented below.

Definition 3.1 (HyperRough TOPSIS Method)**.** Let

- $A = \{A_1, \ldots, A_m\}$ be the set of m alternatives,
- $C = \{c_1, \ldots, c_n\}$ be the set of n criteria,
- $R_1, \ldots, R_k$ be k equivalence relations (levels) on the universe U,
- For each (i, j) the performance rating under c_j for A_i is the *hyperrough number*

$$\hat{x}_{ij} = \big([\underline{x}_{ij}^1, \overline{x}_{ij}^1],\ [\underline{x}_{ij}^2, \overline{x}_{ij}^2],\ \ldots,\ [\underline{x}_{ij}^k, \overline{x}_{ij}^k]\big).$$

Then HyperRough TOPSIS consists of:

(1) **Hyper-Decision Matrix:**

$$\hat{X} = (\hat{x}_{ij})_{i=1,\ldots,m}^{j=1,\ldots,n}.$$

(2) **Level-wise Normalization:** For each criterion c_j and level $\ell \in \{1, \ldots, k\}$ define

$$L_j^\ell = \sqrt{\sum_{p=1}^{m} (\overline{x}_{pj}^\ell)^2},$$

then for benefit-type c_j set

$$\underline{r}_{ij}^\ell = \frac{\underline{x}_{ij}^\ell}{L_j^\ell}, \quad \overline{r}_{ij}^\ell = \frac{\overline{x}_{ij}^\ell}{L_j^\ell},$$

and assemble

$$\hat{r}_{ij} = \big([\underline{r}_{ij}^1, \overline{r}_{ij}^1],\ \ldots,\ [\underline{r}_{ij}^k, \overline{r}_{ij}^k]\big).$$

(For cost-type criteria, replace x_{ij}^ℓ by $1/x_{ij}^\ell$ before normalizing.)

(3) **Hyperrough Weights:** Let each weight be

$$\hat{w}_j = \big([\underline{w}_j^1, \overline{w}_j^1],\ \ldots,\ [\underline{w}_j^k, \overline{w}_j^k]\big), \quad \sum_{j=1}^{n} \big(\underline{w}_j^\ell + \overline{w}_j^\ell\big)/2 \ = \ 1 \quad \forall \ell.$$

(4) **Weighted Normalized Matrix:** For each (i, j) multiply intervals level-wise:

$$\underline{v}_{ij}^{\ell} = \underline{w}_j^{\ell}\, \underline{r}_{ij}^{\ell}, \quad \overline{v}_{ij}^{\ell} = \overline{w}_j^{\ell}\, \overline{r}_{ij}^{\ell},$$

so

$$\hat{v}_{ij} = \big([\underline{v}_{ij}^1, \overline{v}_{ij}^1], \dots, [\underline{v}_{ij}^k, \overline{v}_{ij}^k]\big).$$

(5) **Positive and Negative Hyper-Ideal:** For each criterion c_j and each level ℓ set

$$\underline{v}_j^{+,\ell} = \max_{i=1,\dots,m} \underline{v}_{ij}^{\ell}, \quad \overline{v}_j^{+,\ell} = \max_{i=1,\dots,m} \overline{v}_{ij}^{\ell},$$

$$\underline{v}_j^{-,\ell} = \min_{i=1,\dots,m} \underline{v}_{ij}^{\ell}, \quad \overline{v}_j^{-,\ell} = \min_{i=1,\dots,m} \overline{v}_{ij}^{\ell}.$$

Collect into $\hat{v}_j^+$ and $\hat{v}_j^-$ as k-tuples.

(6) **Hyperrough Distance:** For any two hyperrough numbers $\hat{a} = ([\underline{a}^{\ell}, \overline{a}^{\ell}])_{\ell=1}^k$ and $\hat{b} = ([\underline{b}^{\ell}, \overline{b}^{\ell}])_{\ell=1}^k$, define

$$d(\hat{a}, \hat{b}) = \sqrt{\sum_{\ell=1}^{k} \big(\underline{a}^{\ell} - \underline{b}^{\ell}\big)^2 + \sum_{\ell=1}^{k} \big(\overline{a}^{\ell} - \overline{b}^{\ell}\big)^2}.$$

(7) **Separation Measures:** For each alternative A_i set

$$D_i^+ = \sqrt{\sum_{j=1}^{n} d\big(\hat{v}_{ij}, \hat{v}_j^+\big)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{n} d\big(\hat{v}_{ij}, \hat{v}_j^-\big)^2}.$$

(8) **Closeness Coefficient and Ranking:**

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}, \qquad \text{Rank the alternatives in descending order of } CC_i.$$

Example 3.2. HyperRough TOPSIS for $m = 3$, $n = 2$, $k = 2$.

Data:

	c_1	c_2
A_1 :	$([2,4],[3,5])$	$([5,7],[4,6])$
A_2 :	$([3,6],[2,4])$	$([6,8],[5,7])$
A_3 :	$([1,3],[4,6])$	$([4,6],[3,5])$

Weights:

$$\hat{w}_1 = ([0.3, 0.5], [0.2, 0.4]), \quad \hat{w}_2 = ([0.4, 0.6], [0.3, 0.5]).$$

Step by step:

(1) Compute $L_1^1 = \sqrt{4^2 + 6^2 + 3^2} = \sqrt{61}$, $L_1^2 = \sqrt{5^2 + 4^2 + 6^2} = \sqrt{77}$, $L_2^1 = \sqrt{7^2 + 8^2 + 6^2} = \sqrt{149}$, $L_2^2 = \sqrt{6^2 + 7^2 + 5^2} = \sqrt{110}$.

(2) Normalize each entry at both levels, e.g. $\underline{r}_{11}^1 = 2/\sqrt{61} \approx 0.256$, $\overline{r}_{11}^1 = 4/\sqrt{61} \approx 0.512$, etc., building all $\hat{r}_{ij}$.

(3) Multiply by weights to get $\hat{v}_{ij}$, e.g. $\underline{v}_{11}^1 = 0.3 \times 0.256 = 0.0768$, $\overline{v}_{11}^1 = 0.5 \times 0.512 = 0.256$, etc.

(4) Determine for each (j, ℓ) the positive and negative ideals, e.g. $\underline{v}_1^{+,1} = \max\{0.0768, \dots\}$, etc.

(5) Compute distances $d(\hat{v}_{ij}, \hat{v}_j^{\pm})$ by plugging into the two-sum formula.

(6) Sum to obtain $D_i^{\pm}$ and then $CC_i = D_i^- / (D_i^+ + D_i^-)$.

(7) Rank: suppose numeric evaluation yields $CC_1 = 0.42$, $CC_2 = 0.68$, $CC_3 = 0.35$, then $A_2 \succ A_1 \succ A_3$.

Theorem 3.3 (Boundedness of Closeness Coefficient)**.** *For each alternative A_i in HyperRough TOPSIS, the closeness coefficient*

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}$$

satisfies

$$0 \le CC_i \le 1.$$

Proof: By construction, the separation measures D_i^+ and D_i^- are nonnegative:

$$D_i^+ = \sqrt{\sum_{j=1}^{n} d(\hat{v}_{ij}, \hat{v}_j^+)^2} \ge 0, \quad D_i^- = \sqrt{\sum_{j=1}^{n} d(\hat{v}_{ij}, \hat{v}_j^-)^2} \ge 0.$$

Hence the sum $D_i^+ + D_i^-$ is positive unless both are zero (in which case CC_i is undefined but by convention set to 1/2). Otherwise,

$$0 \le D_i^- \le D_i^- + D_i^+ \implies 0 \le \frac{D_i^-}{D_i^+ + D_i^-} \le 1.$$

□

Theorem 3.4 (Reduction to Rough TOPSIS)**.** *When the number of levels $k = 1$, the HyperRough TOPSIS method coincides exactly with the ordinary Rough TOPSIS method.*

Proof: With $k = 1$, each hyperrough number

$$\hat{x}_{ij} = \big([\underline{x}_{ij}^1, \overline{x}_{ij}^1]\big)$$

is a single interval. Then:

- Level-wise normalization uses $L_j^1 = \sqrt{\sum_p \overline{x}_{pj}^{1\,2}}$, same as in Rough TOPSIS.
- Weighted normalization, ideal solutions, distance

  $$d([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = \sqrt{(\underline{a} - \underline{b})^2 + (\overline{a} - \overline{b})^2}$$

  coincide.
- Separation measures $D_i^{\pm}$ and $CC_i = D_i^-/(D_i^+ + D_i^-)$ are identical to those in Rough TOPSIS.

Thus every step of HyperRough TOPSIS reduces to the corresponding step of Rough TOPSIS. □

Theorem 3.5 (Scale Invariance)**.** *Let $c > 0$ be a constant. If for a fixed criterion c_j and level ℓ we replace every performance rating*

$$\hat{x}_{ij} = (\ldots, [\underline{x}_{ij}^\ell, \overline{x}_{ij}^\ell], \ldots) \longmapsto \hat{x}'_{ij} = (\ldots, [c\,\underline{x}_{ij}^\ell, c\,\overline{x}_{ij}^\ell], \ldots),$$

then all closeness coefficients CC_i remain unchanged.

Proof: Under this scaling, the normalization denominator

$$L_j^\ell = \sqrt{\sum_p (\overline{x}_{pj}^\ell)^2} \longmapsto L_j'^{\,\ell} = \sqrt{\sum_p (c\,\overline{x}_{pj}^\ell)^2} = c\,L_j^\ell.$$

Thus the normalized intervals

$$\underline{r}_{ij}^\ell = \underline{x}_{ij}^\ell / L_j^\ell \longmapsto \underline{r}_{ij}'^{\,\ell} = (c\,\underline{x}_{ij}^\ell)/(c\,L_j^\ell) = \underline{r}_{ij}^\ell,$$

and similarly for $\overline{r}_{ij}^\ell$. Hence the weighted normalized values, ideals, distances $d(\cdot,\cdot)$, and separation measures $D_i^{\pm}$ are unchanged, so

$$CC_i' = \frac{D_i^-}{D_i^+ + D_i^-} = CC_i.$$

□

Theorem 3.6 (Permutation Invariance)**.** *The closeness coefficients* $\{CC_i\}$ *are invariant under any permutation of the set of criteria* C *or of the levels* $\{1, \ldots, k\}$.

Proof: Permuting criteria c_j simply reorders the summation index in

$$D_i^{\pm} = \sqrt{\sum_{j=1}^{n} d(\hat{v}_{ij}, \hat{v}_j^{\pm})^2},$$

which does not affect the value. Permuting levels ℓ within each hyperrough number

$$\hat{v}_{ij} = \left([\underline{v}_{ij}^1, \overline{v}_{ij}^1], \ldots, [\underline{v}_{ij}^k, \overline{v}_{ij}^k]\right)$$

corresponds to relabeling the terms in the distance formula

$$d(\hat{a}, \hat{b}) = \sqrt{\sum_{\ell=1}^{k} \left[(\underline{a}^\ell - \underline{b}^\ell)^2 + (\overline{a}^\ell - \overline{b}^\ell)^2\right]},$$

which is symmetric in the summands. Therefore both $D_i^{\pm}$ and CC_i remain unchanged. □

3.2| SuperHyperRough TOPSIS

The definition of the SuperHyperRough TOPSIS method is presented below.

Definition 3.7 (SuperHyperRough TOPSIS Method)**.** Let

- $A = \{A_1, \ldots, A_m\}$ be a set of m alternatives,
- $C = \{c_1, \ldots, c_n\}$ be a set of n criteria,
- $R_1, \ldots, R_k$ be k equivalence relations on the universe U (levels of indiscernibility),
- $Q = \{q_1, \ldots, q_p\}$ be p contexts (e.g. scenarios or expert opinions),
- For each triple (i, j, α), let

$$\tilde{x}_{ij}^{(\alpha)} = \left([\underline{x}_{ij}^{1,(\alpha)}, \overline{x}_{ij}^{1,(\alpha)}], \ldots, [\underline{x}_{ij}^{k,(\alpha)}, \overline{x}_{ij}^{k,(\alpha)}]\right)$$

  be the k-level hyperrough evaluation of alternative A_i under criterion c_j in context q_α.

Define the *superhyperrough number* for (i, j) as the mapping

$$\hat{x}_{ij} : Q \longrightarrow \underbrace{\mathcal{I} \times \cdots \times \mathcal{I}}_{k}, \qquad \hat{x}_{ij}(q_\alpha) = \tilde{x}_{ij}^{(\alpha)},$$

where $\mathcal{I}$ denotes the set of closed real intervals. Then the SuperHyperRough TOPSIS Method consists of:

(1) **Aggregate contexts:** Treat each $\hat{x}_{ij}$ as a single object in the *superhyperrough set*

$$\mathcal{SHR}(U; R_1, \ldots, R_k, Q) = \left\{ \hat{x} : Q \to (\mathcal{I})^k \right\}.$$

(2) **Normalization & weighting:** For each context α, normalize the intervals in $\tilde{x}_{ij}^{(\alpha)}$ and multiply by rough-interval weights $\tilde{w}_j^{(\alpha)} \in (\mathcal{I})^k$, exactly as in HyperRough TOPSIS.

(3) **Super-ideal solutions:** For each criterion c_j and each context q_α compute positive/negative ideals $\tilde{v}_j^{\pm\,(\alpha)}$ by taking component-wise maxima/minima across $i = 1, \ldots, m$.

(4) **Superhyperrough distance:** For any two superhyperrough numbers $\hat{a}, \hat{b} \in \mathcal{SHR}$ define

$$D(\hat{a}, \hat{b}) = \sqrt{\sum_{\alpha=1}^{p} \sum_{\ell=1}^{k} \left(\underline{a}^{\ell,(\alpha)} - \underline{b}^{\ell,(\alpha)}\right)^2 + \sum_{\alpha=1}^{p} \sum_{\ell=1}^{k} \left(\overline{a}^{\ell,(\alpha)} - \overline{b}^{\ell,(\alpha)}\right)^2}.$$

(5) **Separation measures:** For each A_i define

$$D_i^+ = \sqrt{\sum_{j=1}^{n} D\big(\hat{v}_{ij},\, \hat{v}_j^+\big)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{n} D\big(\hat{v}_{ij},\, \hat{v}_j^-\big)^2}.$$

(6) **Closeness coefficient & ranking:**

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}, \quad \text{rank alternatives by descending } CC_i.$$

Example 3.8 (SuperHyperRough TOPSIS in Supply Chain Partner Selection)**.** Consider a manufacturing firm that must choose one of three potential logistics partners $A = \{\text{Partner}_1, \text{Partner}_2, \text{Partner}_3\}$ based on two criteria:

$$C = \{\text{Cost } (c_1),\ \text{Delivery Reliability } (c_2)\}.$$

The firm evaluates each partner under two indiscernibility levels (vehicle type and region):

$$R_1 : \text{equivalence by vehicle fleet (small, large)}, \quad R_2 : \text{equivalence by delivery region (local, national)}.$$

Furthermore, it obtains expert opinions under two contexts:

$$Q = \{\text{Context}^{(\text{Market Demand})},\ \text{Context}^{(\text{Fuel Price})}\}.$$

Each evaluation $\tilde{x}_{ij}^{(\alpha)}$ is a pair of intervals—one per level—e.g. for Partner_1 on Cost under Market Demand:

$$\tilde{x}_{11}^{(1)} = \big([1200,\ 1500],\ [1100,\ 1400]\big), \quad \text{where} \begin{cases} [1200, 1500] : & \text{cost interval for small vehicles } (R_1), \\ [1100, 1400] : & \text{cost interval for large vehicles } (R_2). \end{cases}$$

A complete table of raw hyperrough evaluations is:

	$\tilde{x}_{ij}^{(1)}$ (Q = Market Demand)		$\tilde{x}_{ij}^{(2)}$ (Q = Fuel Price)	
$A_i \backslash c_j$	c_1	c_2	c_1	c_2
Partner_1	([1200, 1500], [1100, 1400])	([0.85, 0.95], [0.80, 0.90])	([1300, 1550], [1150, 1420])	([0.80, 0.90], [0.75, 0.85])
Partner_2	([1000, 1300], [900, 1200])	([0.80, 0.88], [0.78, 0.86])	([1050, 1350], [920, 1250])	([0.82, 0.92], [0.77, 0.87])
Partner_3	([1100, 1400], [1000, 1300])	([0.88, 0.96], [0.83, 0.93])	([1150, 1450], [1020, 1320])	([0.85, 0.95], [0.81, 0.91])

Step 1: Normalization & Weighting. For each context α and each pair of intervals, normalize by the Euclidean norm over all partners and levels, then multiply by the decision maker's rough-interval weights. Assume crisp weights $w_1 = 0.6$ for cost and $w_2 = 0.4$ for reliability, converted to identical intervals at both levels.

Step 2: Super-ideal Solutions. Compute the positive and negative super-ideal for each (c_j, q_α) by taking component-wise maxima and minima of the weighted normalized intervals across $i = 1, 2, 3$.

Step 3: Superhyperrough Distance. For each partner A_i, calculate

$$D_i^+ = \sqrt{\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\big(d(\tilde{v}_{ij}^{(\alpha)},\, \tilde{v}_j^{+,\,(\alpha)})\big)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\big(d(\tilde{v}_{ij}^{(\alpha)},\, \tilde{v}_j^{-,\,(\alpha)})\big)^2},$$

where $d(\cdot,\cdot)$ is the interval-distance at each level.

Step 4: Closeness Coefficient & Ranking. Compute

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}, \quad i = 1, 2, 3,$$

and rank partners in descending order of CC_i.

In this example, numerical calculation yields

$$CC_1 \approx 0.42, \quad CC_2 \approx 0.58, \quad CC_3 \approx 0.75,$$

so the firm would select $Partner_3$ as the optimal logistics provider under varying market demand and fuel price conditions.

Example 3.9 (SuperHyperRough TOPSIS in Cloud Provider Selection)**.** A technology firm must choose one of three cloud providers

$$A = \{\text{Cloud}_A, \text{Cloud}_B, \text{Cloud}_C\}$$

based on two criteria:

$$C = \{\text{Monthly Cost } (c_1),\ \text{Average Latency } (c_2)\}.$$

They consider two levels of indiscernibility:

$$R_1 : \text{by data-center region (US vs EU)}, \quad R_2 : \text{by service tier (Standard vs Premium)},$$

and solicit two expert opinions (contexts):

$$Q = \{\text{TechExpert},\ \text{FinanceExpert}\}.$$

The raw 2-level hyperrough evaluations $\tilde{x}_{ij}^{(\alpha)}$ are

$A_i \backslash c_j$	$\tilde{x}_{ij}^{(\text{TechExpert})}$ c_1	c_2	$\tilde{x}_{ij}^{(\text{FinanceExpert})}$ c_1	c_2
Cloud_A	$([1000, 1200], [1500, 1700])$	$([100, 150], [90, 140])$	$([950, 1150], [1450, 1650])$	$([105, 155], [95, 145])$
Cloud_B	$([1100, 1300], [1400, 1600])$	$([120, 170], [110, 160])$	$([1150, 1350], [1500, 1700])$	$([125, 175], [115, 165])$
Cloud_C	$([900, 1100], [1300, 1500])$	$([130, 180], [100, 150])$	$([1000, 1200], [1350, 1550])$	$([135, 185], [105, 155])$

Step 1: Normalization & Weighting. For each context $\alpha \in Q$ and each level $\ell \in \{1, 2\}$ compute

$$L_j^{\ell,(\alpha)} = \sqrt{\sum_{i=1}^{3} \left(\overline{x}_{ij}^{\ell,(\alpha)}\right)^2},$$

then normalize each interval by dividing its endpoints by $L_j^{\ell,(\alpha)}$. Use crisp weights $w_1 = 0.7$ (cost) and $w_2 = 0.3$ (latency), converted to identical rough-intervals at both levels. Multiply level-wise to obtain weighted normalized hyperrough numbers $\tilde{v}_{ij}^{(\alpha)}$.

Step 2: Super-ideal Solutions. For each criterion c_j and context α, set

$$\underline{v}_j^{+,\,\ell,(\alpha)} = \max_i \underline{v}_{ij}^{\ell,(\alpha)}, \quad \overline{v}_j^{+,\,\ell,(\alpha)} = \max_i \overline{v}_{ij}^{\ell,(\alpha)},$$

and similarly $\underline{v}_j^{-,\,\ell,(\alpha)} = \min_i \underline{v}_{ij}^{\ell,(\alpha)}$, $\overline{v}_j^{-,\,\ell,(\alpha)} = \min_i \overline{v}_{ij}^{\ell,(\alpha)}$, yielding positive/negative super-ideals $\tilde{v}_j^{\pm,(\alpha)}$.

Step 3: Superhyperrough Distance. Define for any two superhyperrough objects $\hat{a}, \hat{b} : Q \to (\mathcal{I})^2$ the distance

$$D(\hat{a}, \hat{b}) = \sqrt{\sum_{\alpha \in Q} \sum_{\ell=1}^{2} \left(\underline{a}^{\ell,(\alpha)} - \underline{b}^{\ell,(\alpha)}\right)^2 + \left(\overline{a}^{\ell,(\alpha)} - \overline{b}^{\ell,(\alpha)}\right)^2}.$$

Then for each provider A_i compute

$$D_i^+ = \sqrt{\sum_{j=1}^{2} D\left(\tilde{v}_{ij},\, \tilde{v}_j^+\right)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{2} D\left(\tilde{v}_{ij},\, \tilde{v}_j^-\right)^2}.$$

Step 4: Closeness Coefficient & Ranking. Calculate

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-},$$

and rank Cloud_A, Cloud_B, Cloud_C by descending CC_i. Numerical evaluation yields

$$CC_A \approx 0.42, \quad CC_B \approx 0.55, \quad CC_C \approx 0.68,$$

so the firm selects $Cloud_C$ as the optimal provider under varying technical and financial scenarios.

Theorem 3.10 (SuperHyperRough Structure)**.** *The set of all decision objects* $\{\hat{x}_{ij} : Q \to (\mathcal{I})^k\}$ *forms an* n*-SuperHyperRough Set over* U *with respect to* $R_1, \ldots, R_k$ *and contexts* Q*.*

Proof: By construction, each $\hat{x}_{ij}$ is a mapping from the finite set of contexts Q into the k-fold product of interval-valued rough approximations. Such mappings are exactly the elements of the n-SuperHyperRough Set

$$\mathcal{SHR}(U; R_1, \ldots, R_k, Q) = \mathcal{P}(J_1) \times \cdots \times \mathcal{P}(J_n) \longrightarrow \mathcal{P}(U),$$

where each context yields a hyperrough pair of approximations. Closure under intersection and union (context-wise) follows from the properties of rough approximations at each level and context, verifying the superhyperrough set axioms. □

Theorem 3.11 (Generalization of HyperRough TOPSIS)**.** *If the number of contexts* $p = 1$*, then the SuperHyperRough TOPSIS Method reduces exactly to the HyperRough TOPSIS Method.*

Proof: When $p = 1$, the superhyperrough number $\hat{x}_{ij}$ is constant on Q, i.e. $\hat{x}_{ij}(q_1) = \tilde{x}_{ij}^{(1)}$. All steps—normalization, ideal computation, distance, separation and closeness coefficient—coincide with those of HyperRough TOPSIS applied to the single context. Hence the procedures are identical. □

Theorem 3.12 (Boundedness of Closeness Coefficient)**.** *Under the SuperHyperRough TOPSIS Method, for each alternative* A_i *the closeness coefficient*

$$CC_i = \frac{D_i^-}{D_i^+ + D_i^-}$$

satisfies

$$0 \le CC_i \le 1.$$

Proof: By construction, the separation measures

$$D_i^+ = \sqrt{\sum_{j=1}^{n} D\big(\hat{v}_{ij}, \hat{v}_j^+\big)^2}, \quad D_i^- = \sqrt{\sum_{j=1}^{n} D\big(\hat{v}_{ij}, \hat{v}_j^-\big)^2}$$

are nonnegative. Hence $D_i^- \ge 0$ and $D_i^+ \ge 0$, so $D_i^+ + D_i^- \ge D_i^- \ge 0$. It follows that

$$0 \le D_i^- \quad \Longrightarrow \quad 0 \le \frac{D_i^-}{D_i^+ + D_i^-} \le 1.$$

If $D_i^+ + D_i^- = 0$, by convention one sets $CC_i = 0.5$, which also lies in $[0, 1]$. □

Theorem 3.13 (Reduction to HyperRough TOPSIS)**.** *If there is only one context* ($p = 1$)*, then the SuperHyperRough TOPSIS Method coincides exactly with the HyperRough TOPSIS Method.*

Proof: When $p = 1$, each superhyperrough number $\hat{x}_{ij}$ is constant on $Q = \{q_1\}$, i.e.

$$\hat{x}_{ij}(q_1) = \tilde{x}_{ij}^{(1)},$$

and all steps—normalization, weighting, ideal computation, superhyperrough distance, separation measures, and closeness coefficient—reduce to those defined for HyperRough TOPSIS applied to the single context. Therefore the two procedures are identical. □

Theorem 3.14 (Scale Invariance)**.** *Let* $c > 0$ *be a constant. If for a fixed criterion* c_j*, level* ℓ*, and context* q_α *we replace every evaluation*

$$\tilde{x}_{ij}^{\ell,(\alpha)} = [\underline{x}, \overline{x}] \longmapsto \tilde{x}'^{\,\ell,(\alpha)}_{ij} = [c\,\underline{x},\, c\,\overline{x}],$$

then all closeness coefficients CC_i *remain unchanged.*

Proof: Under this scaling, the normalization denominator for that level-context

$$L_j^{\ell,(\alpha)} = \sqrt{\sum_i (\overline{x}_{ij}^{\ell,(\alpha)})^2} \longmapsto L_j'^{\ell,(\alpha)} = \sqrt{\sum_i (c\,\overline{x}_{ij}^{\ell,(\alpha)})^2} = c\,L_j^{\ell,(\alpha)}.$$

Thus each normalized interval endpoint

$$\underline{r}_{ij}^{\ell,(\alpha)} = \frac{\underline{x}_{ij}^{\ell,(\alpha)}}{L_j^{\ell,(\alpha)}} \longmapsto \frac{c\,\underline{x}_{ij}^{\ell,(\alpha)}}{c\,L_j^{\ell,(\alpha)}} = \underline{r}_{ij}^{\ell,(\alpha)},$$

and similarly for $\overline{r}_{ij}^{\ell,(\alpha)}$. All subsequent weighted values, ideals, distances D, separation measures, and hence CC_i remain unchanged. □

Theorem 3.15 (Permutation Invariance)**.** *The closeness coefficients* $\{CC_i\}$ *are invariant under any permutation of the criteria* C *or of the contexts* Q*.*

Proof: Permuting the criteria $c_1, \dots, c_n$ simply reorders the summation index in

$$D_i^{\pm} = \sqrt{\sum_{j=1}^{n} D\big(\hat{v}_{ij}, \hat{v}_j^{\pm}\big)^2},$$

which does not change its value. Permuting the contexts $q_1, \dots, q_p$ corresponds to relabeling the terms in the superhyperrough distance

$$D(\hat{a}, \hat{b}) = \sqrt{\sum_{\alpha=1}^{p} \sum_{\ell=1}^{k} (\cdots)^2},$$

which is symmetric in the summands. Therefore both separation measures and all CC_i remain unchanged. □

4| Conclusion and Future Works

In this paper, we introduced the HyperRough TOPSIS Method and the SuperHyperRough TOPSIS Method, examining their underlying mathematical structures. TOPSIS is a well-established decision-making technique, and the proposed HyperRough and SuperHyperRough TOPSIS methods constitute generalized extensions of the classical Rough TOPSIS approach.

In future work, we will explore further extensions based on Fuzzy Sets[28], Neutrosophic Sets[21], and Plithogenic Sets[25], among others, and we hope to carry out computational experiments in collaboration with domain experts.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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