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Convexity Cum Concavity on Refined Fuzzy Set with Some Properties

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Abstract

Smarandache refined fuzzy sets to handle an object's sub-membership degrees. Applications for fuzzy convexity are numerous and include pattern recognition, optimization, and related issues. By taking into account a more precise definition of fuzzy sets, these applications can be handled more effectively. This paper uses theoretical and analytical techniques to construct the novel idea of convexity cum concavity on refined fuzzy sets. The convex (concave) fuzzy sets proposed by Zadeh [1], [2] and Chaudhuri [3], [4] are extended in this work. Some of its significant findings are also generalizable.

Keywords: Refined fuzzy set, Concave refined fuzzy set, Convex refined fuzzy set, Ortho-concave refined fuzzy set, Ortho-convex refined fuzzy set.

1 | Introduction

The fuzzy set notion, which forms the foundation of the theory of possibility, was first presented by Zadeh [1], [2]. This fuzzy set notion was expanded upon by Dubois and Prade [3], and it was used in the analysis of differential gene expression data by Liang et al. [4]. Similarity metrics for fuzzy sets were later established by Beg and Ashraf [5]. Set difference and symmetric difference of fuzzy sets were described by Vemuri et al. [6]. The work was expanded to include the complement of an extended fuzzy set by Neog and Sut [7]. Yager [8] created Pythagorean fuzzy subsets, talked about their characteristics and uses, and then applied them to multi-criteria decision making. Biswas [9], McBratney and Odeh [10] and Mamdani [11] applied fuzzy sets in students' evaluation, soil science, and control of simple dynamic plants.

Chaudhuri [3], [4] presented the idea of a concave fuzzy set, talked about some of its practical characteristics, and explained some related ideas and their computing methods. The development of fuzzy geometry and fuzzy structures benefits from this idea. This idea was expanded to include convex and concave fuzzy mappings by Syau [14]. Concavo-convex fuzzy sets were introduced by Sarkar [15], who also established some

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intriguing characteristics of this unique kind of fuzzy set. Concave type-2 fuzzy sets were created by Tahayori et al. [16], who also constructed and validated its operations and attributes. Measures of fuzziness for concave functions and measures of fuzzy sets were described by Weber [17]. Fuzzy uncertainty measurement for monotone rising concave functions was described by Pal and Bezdek [18].

The concept of fuzzy sets was improved by Smarandache [19], who created refined ternary fuzzy sets, refined picture fuzzy sets, refined intuitionistic fuzzy sets, and refined intuitionistic fuzzy sets. More refinements in fuzzy sets, including refined Pythagorean fuzzy sets, refined Atanassov's intuitionistic fuzzy sets of type 2, refined spherical fuzzy sets, refined n-hyper spherical fuzzy sets, refined q-rung orthopair fuzzy sets, etc., were made possible by this idea of refinement.

This study develops a new concept of convexity cum concavity on refined fuzzy sets and extends the concept of convexity cum concavity on fuzzy sets. Furthermore, in this context, certain significant traits and outcomes are highlighted.

The remainder of the paper is structured as follows. Section 2 reviews the fundamental definitions and terminology used in the literature in relation to the main finding; Section 3 introduces the novel idea of convexity cum concavity on refined fuzzy sets along with various properties and results; and Section 4 wraps up the paper.

2 | Preliminaries

In this section, some basic definitions and terms are recalled from the literature to support the main study of this work. Here, Z, G, and I will play the role of the universal set, R^n and [0, 1], respectively.

Definition 1 ([1]). Set A is a fuzzy set if A is characterized by a membership function Γ_A and defined

$$A = \{(\Gamma_A(\alpha)), \alpha \in Z\}, \Gamma_A: Z \to P(I),$$

where $A \subseteq Z$.

The union of two fuzzy sets A and B with respective membership functions $f_A(\alpha)$ and $f_B(\alpha)$ is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to those of A and B by

$$f_C(\alpha) = \mathrm{Max}(f_A(\alpha), f_B(\alpha)), \qquad \alpha \in Z,$$

or in abbreviated form

$$f_C = f_A \vee f_B.$$

The intersection of two fuzzy sets A and B with respective membership functions $f_A(\alpha)$ and $f_B(\alpha)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to those of A and B by

$$f_C(\alpha) = \mathrm{Min}(f_A(\alpha), f_B(\alpha)), \qquad \alpha \in Z,$$

or in abbreviated form

$$f_C = f_A \wedge f_B.$$

Definition 2 ([1]). A fuzzy set A is convex iff

$$f_A(\lambda\alpha_1 + (1-\lambda)\alpha_2) \geq \mathrm{Min}(f_A(\alpha_1), f_A(\alpha_2)), \qquad \text{for all } \alpha_1, \alpha_2 \in Z \text{ and } \lambda \in I.$$

Definition 3 ([1]). A fuzzy set A is concave iff

$$f_A(\lambda\alpha_1 + (1-\lambda)\alpha_2) \leq \mathrm{Max}(f_A(\alpha_1), f_A(\alpha_2)), \qquad \text{for all } \alpha_1, \alpha_2 \in Z \text{ and } \lambda \in I.$$

Definition 4 ([19]). A refined fuzzy set A is defined as

$$A_{RFS} = \{(\Gamma_A^1(\alpha), \Gamma_A^2(\alpha), \ldots, \Gamma_A^p(\alpha), p \geq 2, \alpha \in A)\},$$

where Γ_A^j is sub-membership of degree j^{th}-type of elements of Z w.r.t. A, and is subset of I for $1 \leq j \leq p$ and $\sum_{j=1}^{p} \sup \Gamma_\alpha^j \leq 1$, for all $\alpha \in A$.

3 | Convex and Concave Refined Fuzzy Sets

In this section, convex and concave refined fuzzy sets are defined. Some important results are discussed.

Definition 5. A refined fuzzy set A_{RFS} in G is Convex if for all $u, v \in G$ and all w on the line segment $\overline{uv}$

$$\Gamma_{A_{RFS}}^k(w) \geq \min\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v)\right), \qquad 1 \leq k \leq n,$$

where $\Gamma_{A_{RFS}}^k$ is sub-membership of degree k^{th}-type of the elements w.r.t. A and is subset of I for $1 \leq k \leq p$ and $\sum_{k=1}^{p} \sup \Gamma^k \leq 1$.

Definition 6. A refined fuzzy set A_{RFS} in G is ortho-convex if for all $u, v \in G$ and all w on the line segment $\overline{uv}$ lie on a line that is parallel to the co-ordinate axis

$$\Gamma_{A_{RFS}}^j(w') \geq \min\left(\Gamma_{A_{RFS}}^j(u'), \Gamma_{A_{RFS}}^j(v')\right), \qquad 1 \leq j \leq n.$$

Remark 1. An ortho-convex RFS is convex RFS, but its converse may or may not be true.

Definition 7. A refined fuzzy set A_{RFS} in G is concave if for all $u, v \in G$ and all w on the line segment $\overline{uv}$

$$\Gamma_{A_{RFS}}^k(w) \leq \max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v)\right), \qquad 1 \leq k \leq n.$$

Definition 8. A refined fuzzy set A_{RFS} in G is ortho-concave if for all $u, v \in G$ and all w on the line segment $\overline{uv}$ lie on line which is parallel to co-ordinate axis

$$\Gamma_{A_{RFS}}^j(w') \leq \max\left(\Gamma_{A_{RFS}}^j(u'), \Gamma_{A_{RFS}}^j(v')\right), \qquad 1 \leq j \leq n.$$

Remark 2. An ortho-concave RFS is a concave RFS, but its converse may or may not be true.

Theorem 1. The complement of convex A RFS is concave RFS.

Proof: if A is convex refined fuzzy then for any two points u and v and another point w which lies on $\overline{uv}$

$$\Gamma_{A_{RFS}}^k(w) \geq \min\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v)\right), \qquad 1 \leq k \leq n,$$

so it becomes

$$\bar{\Gamma}_A^k(w) \leq 1 - \min\left(1 - \bar{\Gamma}_A^k(u), 1 - \bar{\Gamma}_A^k(v)\right), \qquad 1 \leq k \leq n. \quad \textbf{(1)}$$

Now if

$$1 - \bar{\Gamma}_A^k(u) \leq 1 - \bar{\Gamma}_A^k(v),$$

then

$$\min\left(1 - \bar{\Gamma}_A^k(u), 1 - \bar{\Gamma}_A^k(v)\right) = 1 - \bar{\Gamma}_A^k(u),$$

and *Eq. (1)* takes the form

$$\bar{\Gamma}^k_A(w) \le \bar{\Gamma}^k_A(u). \tag{2}$$

Similarly if

$$1-\bar{\Gamma}^k_A(v) \le 1-\bar{\Gamma}^k_A(u),$$

then

$$\min\left(1-\bar{\Gamma}^k_A(u), 1-\bar{\Gamma}^k_A(v)\right) = 1-\bar{\Gamma}^k_A(v),$$

so *Eq. (1)* reduces to

$$\bar{\Gamma}^k_A(w) \le \bar{\Gamma}^k_A(v). \tag{3}$$

Using *Eqs. (2)* and *(3)*, it becomes

$$\bar{\Gamma}^k_A(w) \le \max\left(\bar{\Gamma}^k_A(u), \bar{\Gamma}^k_A(v)\right), \qquad 1 \le k \le n,$$

which means compliment of A_{RFS} is concave RFS.

Remark 3. The complement of ortho-convex A_{RFS} is ortho-concave and hence concave RFS.

Theorem 2. The union of two convex refined fuzzy sets is a convex refined fuzzy set.

Proof: let A_{RFS} and B_{RFS} be two convex refined fuzzy sets and $H = A_{RFS} \cup B_{RFS}$.

Consider two points u and v and another point w which lies on $\overline{uv}$.

Now

$$\Gamma^k_{H_{RFS}}(u) = \min\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{B_{RFS}}(u)\right), \qquad 1 \le k \le n, \tag{4}$$

$$\Gamma^k_{H_{RFS}}(v) = \min\left(\Gamma^k_{A_{RFS}}(v), \Gamma^k_{B_{RFS}}(v)\right), \qquad 1 \le k \le n, \tag{5}$$

$$\Gamma^k_{H_{RFS}}(w) = \min\left(\Gamma^k_{A_{RFS}}(w), \Gamma^k_{B_{RFS}}(w)\right), \qquad 1 \le k \le n. \tag{6}$$

Now

$$\min\left(\Gamma^k_{H_{RFS}}(u), \Gamma^k_{H_{RFS}}(v)\right)$$

$$= \min\left(\min\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{B_{RFS}}(u)\right), \min\left(\Gamma^k_{A_{RFS}}(v), \Gamma^k_{B_{RFS}}(v)\right)\right) \tag{7}$$

$$= \min\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{B_{RFS}}(u), \Gamma^k_{A_{RFS}}(v), \Gamma^k_{B_{RFS}}(v)\right).$$

Let $\Gamma^k_{A_{RFS}}(w) \le \Gamma^k_{B_{RFS}}(w)$ so *Eq. (7)* becomes

$$\Gamma^k_{H_{RFS}}(w) = \Gamma^k_{A_{RFS}}(w),$$

as A is refined convex fuzzy set so

$$\Gamma^k_{A_{RFS}}(w) \ge \min\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{A_{RFS}}(v)\right) \ge \min\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{B_{RFS}}(u), \Gamma^k_{A_{RFS}}(v), \Gamma^k_{B_{RFS}}(v)\right),$$

i.e.

$$\Gamma^{k}_{A_{RFS}}(w) = \Gamma^{k}_{H_{RFS}}(w) \geq \min\left(\Gamma^{k}_{H_{RFS}}(u)\Gamma^{k}_{H_{RFS}}(v)\right).$$

Similarly for $\Gamma^{k}_{B_{RFS}}(w) \leq \Gamma^{k}_{A_{RFS}}(w)$, *Eq. (6)* takes the form

$$\Gamma^{k}_{H_{RFS}}(w) = \Gamma^{k}_{B_{RFS}}(w), \tag{8}$$

as B is a refined convex fuzzy set, so *Eq. (8)* becomes

$$\Gamma^{k}_{B_{RFS}}(w) \geq \min\left(\Gamma^{k}_{B_{RFS}}(u), \Gamma^{k}_{B_{RFS}}(v)\right) \geq \min\left(\Gamma^{k}_{A_{RFS}}(u), \Gamma^{k}_{B_{RFS}}(u), \Gamma^{k}_{A_{RFS}}(v), \Gamma^{k}_{B_{RFS}}(v)\right),$$

i.e.

$$\Gamma^{k}_{H_{RFS}}(w) \geq \min\left(\Gamma^{k}_{H_{RFS}}(u)\Gamma^{k}_{H_{RFS}}(v)\right).$$

Hence, the proof is complete.

Theorem 3. The union of two ortho-convex RFS is an ortho-convex RFS and hence convex RFS.

Proof: let A_{RFS} and B_{RFS} be two convex refined fuzzy sets and $H = A_{RFS} \cup B_{RFS}$.

Consider two points u' and v' and another point w' which lies on $\overline{u'v'}$, which is parallel to coordinate axis.

Now

$$\Gamma^{k}_{H_{RFS}}(u') = \min\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u')\right), \qquad 1 \leq k \leq n, \tag{9}$$

$$\Gamma^{k}_{H_{RFS}}(v') = \min\left(\Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right), \qquad 1 \leq k \leq n, \tag{10}$$

$$\Gamma^{k}_{H_{RFS}}(w') = \min\left(\Gamma^{k}_{A_{RFS}}(w'), \Gamma^{k}_{B_{RFS}}(w')\right), \qquad 1 \leq k \leq n. \tag{11}$$

Take

$$\min\left(\Gamma^{k}_{H_{RFS}}(u'), \Gamma^{k}_{H_{RFS}}(v')\right)$$

$$= \min\left(\min\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u')\right), \min\left(\Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right)\right) \tag{12}$$

$$= \min\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right).$$

Let $\Gamma^{k}_{A_{RFS}}(w') \leq \Gamma^{k}_{B_{RFS}}(w')$ in *Eq. (12)* so that

$$\Gamma^{k}_{H_{RFS}}(w') = \Gamma^{k}_{A_{RFS}}(w'),$$

as A is ortho-refined convex fuzzy set so

$$\Gamma^{k}_{A_{RFS}}(w') \geq \min\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v')\right) \geq \min\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right),$$

i.e.

$$\Gamma^k_{A_{RFS}}(w') = \Gamma^k_{H_{RFS}}(w') \geq \min\left(\Gamma^k_{H_{RFS}}(u')\Gamma^k_{H_{RFS}}(v')\right).$$

Similarly for $\Gamma^k_{B_{RFS}}(w') \leq \Gamma^k_{A_{RFS}}(w')$, *Eq. (11)* takes the form

$$\Gamma^k_{H_{RFS}}(w') = \Gamma^k_{B_{RFS}}(w'),$$

as B is an ortho-refined convex fuzzy set, so

$$\Gamma^k_{B_{RFS}}(w') \geq \min\left(\Gamma^k_{B_{RFS}}(u'), \Gamma^k_{B_{RFS}}(v')\right) \geq \min\left(\Gamma^k_{A_{RFS}}(u'), \Gamma^k_{B_{RFS}}(u'), \Gamma^k_{A_{RFS}}(v'), \Gamma^k_{B_{RFS}}(v')\right),$$

i.e.

$$\Gamma^k_{B_{RFS}}(w') = \Gamma^k_{H_{RFS}}(w') \geq \min\left(\Gamma^k_{H_{RFS}}(u')\Gamma^k_{H_{RFS}}(v')\right).$$

Since every ortho-convex refined fuzzy set is also a convex refined fuzzy set which leads to the completion of proof.

Remark 4. The union of a family of convex refined fuzzy sets is a convex refined fuzzy set.

Remark 5. The union of the family of ortho-convex refined fuzzy sets is ortho-convex refined fuzzy, and hence convex refined fuzzy set.

Theorem 4. The complement of concave A_{RFS} is convex RFS.

Proof: if A is concave refined fuzzy, then for any two points u and v and another point w which lies on $\overline{uv}$.

$$\Gamma^k_{A_{RFS}}(w) \leq \max\left(\Gamma^k_{A_{RFS}}(u), \Gamma^k_{A_{RFS}}(v)\right), \qquad 1 \leq k \leq n,$$

so it becomes

$$\bar{\Gamma}^k_A(w) \geq 1 - \max\left(1 - \bar{\Gamma}^k_A(u), 1 - \bar{\Gamma}^k_A(v)\right), \qquad 1 \leq k \leq n. \tag{13}$$

Now if

$$1 - \bar{\Gamma}^k_A(u) \leq 1 - \bar{\Gamma}^k_A(v),$$

then

$$\max\left(1 - \bar{\Gamma}^k_A(u), 1 - \bar{\Gamma}^k_A(v)\right) = 1 - \bar{\Gamma}^k_A(v),$$

Eq. (13) takes the form

$$\bar{\Gamma}^k_A(w) \geq \bar{\Gamma}^k_A(v). \tag{14}$$

Similarly if

$$1 - \bar{\Gamma}^k_A(v) \leq 1 - \bar{\Gamma}^k_A(u),$$

then

$$\max\left(1 - \bar{\Gamma}^k_A(u), 1 - \bar{\Gamma}^k_A(v)\right) = 1 - \bar{\Gamma}^k_A(u),$$

so *Eq. (13)* reduces to

$$\bar{\Gamma}_A^k(w) \geq \bar{\Gamma}_A^k(u). \quad \textbf{(15)}$$

Eqs. (14) and *(15)* results in

$$\bar{\Gamma}_A^k(w) \geq \min\left(\bar{\Gamma}_A^k(u), \bar{\Gamma}_A^k(v)\right), \qquad 1 \leq k \leq n,$$

which means compliment of A_{RFS} is convex RFS.

Remark 6. The complement of ortho-concave A_{RFS} is ortho-convex and hence convex RFS.

Theorem 5. The union of two concave refined fuzzy sets is a concave refined fuzzy set.

Proof: let A_{RFS} and B_{RFS} be two convex refined fuzzy sets and $H = A_{RFS} \cup B_{RFS}$.

Consider two points u and v and another point w which lies on $\overline{uv}$.

Now

$$\Gamma_{H_{RFS}}^k(u) = \max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{B_{RFS}}^k(u)\right), \qquad 1 \leq k \leq n, \quad \textbf{(16)}$$

$$\Gamma_{H_{RFS}}^k(v) = \max\left(\Gamma_{A_{RFS}}^k(v), \Gamma_{B_{RFS}}^k(v)\right), \qquad 1 \leq k \leq n, \quad \textbf{(17)}$$

$$\Gamma_{H_{RFS}}^k(w) = \max\left(\Gamma_{A_{RFS}}^k(w), \Gamma_{B_{RFS}}^k(w)\right), \qquad 1 \leq k \leq n. \quad \textbf{(18)}$$

Consider

$$\max\left(\Gamma_{H_{RFS}}^k(u), \Gamma_{H_{RFS}}^k(v)\right)$$

$$= \max\left(\max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{B_{RFS}}^k(u)\right), \max\left(\Gamma_{A_{RFS}}^k(v), \Gamma_{B_{RFS}}^k(v)\right)\right) \quad \textbf{(19)}$$

$$= \max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{B_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v), \Gamma_{B_{RFS}}^k(v)\right).$$

Using $\Gamma_{A_{RFS}}^k(w) \geq \Gamma_{B_{RFS}}^k(w)$ in *Eq. (18)* reduces it to

$$\Gamma_{H_{RFS}}^k(w) = \Gamma_{A_{RFS}}^k(w), \quad \textbf{(20)}$$

as A is refined concave fuzzy so, *Eq. (20)* becomes

$$\Gamma_{H_{RFS}}^k(w) = \Gamma_{A_{RFS}}^k(w) \leq \max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v)\right),$$

$$\Gamma_{H_{RFS}}^k(w) \leq \max\left(\Gamma_{A_{RFS}}^k(u), \Gamma_{B_{RFS}}^k(u), \Gamma_{A_{RFS}}^k(v)\Gamma_{B_{RFS}}^k(v)\right).$$

i.e.

$$\Gamma_{H_{RFS}}^k(w) \leq \max\left(\Gamma_{H_{RFS}}^k(u)\Gamma_{H_{RFS}}^k(v)\right).$$

Similarly, if $\Gamma_{B_{RFS}}^k(w) \geq \Gamma_{A_{RFS}}^k(w)$, then *Eq. (18)* results in

$$\Gamma_{H_{RFS}}^k(w) = \Gamma_{B_{RFS}}^k(w), \quad \textbf{(21)}$$

as B is a refined concave fuzzy set so, *Eq. (21)* becomes

$$\Gamma_{H_{RFS}}^k(w) = \Gamma_{B_{RFS}}^k(w) \leq \max\left(\Gamma_{B_{RFS}}^k(u), \Gamma_{B_{RFS}}^k(v)\right),$$

$$\Gamma^{k}_{H_{RFS}}(w) \leq \max\left(\Gamma^{k}_{A_{RFS}}(u), \Gamma^{k}_{B_{RFS}}(u), \Gamma^{k}_{A_{RFS}}(v)\Gamma^{k}_{B_{RFS}}(v)\right),$$

i.e.

$$\Gamma^{k}_{H_{RFS}}(w) \leq \max\left(\Gamma^{k}_{H_{RFS}}(u)\Gamma^{k}_{H_{RFS}}(v)\right).$$

Hence, the required result is proved.

Theorem 6. The union of two ortho-concave RFS is an ortho-concave RFS and hence concave RFS.

Proof: let A_{RFS} and B_{RFS} be two convex refined fuzzy sets and $H = A_{RFS} \cup B_{RFS}$. Consider two points u' and v' and another point w' on $\overline{u'v'}$ with condition that $\overline{u'v'}$ is parallel to coordinate axis.

as

$$\Gamma^{k}_{H_{RFS}}(u') = \max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u')\right), \qquad 1 \leq k \leq n, \tag{22}$$

$$\Gamma^{k}_{H_{RFS}}(v') = \max\left(\Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right), \qquad 1 \leq k \leq n, \tag{23}$$

$$\Gamma^{k}_{H_{RFS}}(w') = \max\left(\Gamma^{k}_{A_{RFS}}(w'), \Gamma^{k}_{B_{RFS}}(w')\right), \qquad 1 \leq k \leq n. \tag{24}$$

Now

$$\max\left(\Gamma^{k}_{H_{RFS}}(u'), \Gamma^{k}_{H_{RFS}}(v')\right)$$

$$= \max\left(\max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u')\right), \max\left(\Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right)\right) \tag{25}$$

$$= \max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v'), \Gamma^{k}_{B_{RFS}}(v')\right).$$

Let $\Gamma^{k}_{A_{RFS}}(w') \geq \Gamma^{k}_{B_{RFS}}(w')$ in *Eq. (25)* so that

$$\Gamma^{k}_{H_{RFS}}(w') = \Gamma^{k}_{A_{RFS}}(w'),$$

as A is ortho-concave refined fuzzy, so

$$\Gamma^{k}_{H_{RFS}}(w') = \Gamma^{k}_{A_{RFS}}(w') \leq \max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v')\right),$$

$$\Gamma^{k}_{H_{RFS}}(w') \leq \max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v')\Gamma^{k}_{B_{RFS}}(v')\right),$$

i.e.

$$\Gamma^{k}_{H_{RFS}}(w') \leq \max\left(\Gamma^{k}_{H_{RFS}}(u')\Gamma^{k}_{H_{RFS}}(v')\right).$$

Similarly for $\Gamma^{k}_{B_{RFS}}(w') \geq \Gamma^{k}_{A_{RFS}}(w')$, *Eq. (24)* takes the form

$$\Gamma^{k}_{H_{RFS}}(w') = \Gamma^{k}_{B_{RFS}}(w'), \tag{26}$$

as B is an ortho-refined concave fuzzy set so, *Eq. (26)* becomes

$$\Gamma^{k}_{H_{RFS}}(w') = \Gamma^{k}_{B_{RFS}}(w') \leq \max\left(\Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v')\right),$$

$$\Gamma^{k}_{H_{RFS}}(w') \leq \max\left(\Gamma^{k}_{A_{RFS}}(u'), \Gamma^{k}_{B_{RFS}}(u'), \Gamma^{k}_{A_{RFS}}(v')\Gamma^{k}_{B_{RFS}}(v')\right),$$

i.e.

$$\Gamma^{k}_{H_{RFS}}(w') \le \max\left(\Gamma^{k}_{H_{RFS}}(u')\Gamma^{k}_{H_{RFS}}(v')\right).$$

Since every ortho-concave refined fuzzy set is also a concave refined fuzzy set, which leads to the completion of the proof.

Remark 7. The union of a family of concave refined fuzzy sets is a concave refined fuzzy set.

Remark 8. The union of the family of ortho-concave refined fuzzy sets is ortho-concave refined fuzzy and hence concave refined fuzzy set.

Definition 9. For any point $p \in l$ where l is a line, l_p is perpendicular to l at A, the Inf-Projection A_l of concave refined fuzzy set A in R^2 is mapping of each point $p \in l$ into $\inf\{A(r), r \in l_p\}$.

Definition 10. For any point $p \in l$ where l is a line, l_p is perpendicular to l at A, the Sup-Projection A_l of concave refined fuzzy set A in R^2 is mapping of each point $p \in l$ into $\sup\{A(r), r \in l_p\}$.

Theorem 7. If A is concave RFS, so is A_l.

Proof: if u, v, w are three points of l such that w lies on $\overline{uv}$, given any $\varepsilon > 0$, let u' and v' be points on l_u and l_v so that $\Gamma^{k}_{A_l}(u) > \Gamma^{k}_{A}(u') - \varepsilon$ and $\Gamma^{k}_{A_l}(v) > \Gamma^{k}_{A}(v') - \varepsilon$. Let w' be the intersection of line segment $\overline{u'v'}$ with l_w. Since A is concave and $w' \in \overline{u'v'}$, then we have

$$\begin{aligned}\Gamma^{k}_{A}(w') &\le \max\left(\Gamma^{k}_{A}(u'), \Gamma^{k}_{A}(v')\right), \qquad 1 \le k \le n,\\ &< \max\left(\Gamma^{k}_{A_l}(u) + \epsilon, \Gamma^{k}_{A_l}(v) + \varepsilon\right)\\ &= \max\left(\Gamma^{k}_{A_l}(u), \Gamma^{k}_{A_l}(v)\right) + \varepsilon.\end{aligned}$$

But by the definition of inf-projection

$$\Gamma^{k}_{A}(w') \ge \Gamma^{k}_{A_l}(w).$$

Hence

$$\Gamma^{k}_{A_l}(w) < \max\left(\Gamma^{k}_{A_l}(u), \Gamma^{k}_{A_l}(v)\right) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary so

$$\Gamma^{k}_{A_l}(w) \le \max\left(\Gamma^{k}_{A_l}(u), \Gamma^{k}_{A_l}(v)\right),$$

which means A_l is concave.

Remark 9. If A is convex RFS, so is A_l.

4 | Conclusion

In this paper, convexity cum concavity is defined on a refined fuzzy set, and some useful results are established. This work can further be extended by developing convex hull, convex cone, α-cuts, and other types of convexity like graded convexity, triangular convexity, etc.

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Author Contributaion

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No data is associated with this study.

Conflicts of Interest

The authors declare no conflict of interest.

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