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# Soft Binary Piecewise Plus Operation: A New Type of Operation For Soft Sets

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## Abstract

After being presented by Molodtsov [1], soft set theory became well-known as a cutting-edge method for addressing uncertainty-related issues and modeling uncertainty. It may be used in a range of theoretical and practical applications. The soft binary piecewise plus operation is a novel soft set operation presented in this work. Its fundamental algebraic properties are investigated in detail. Additionally, the distributions of this operation over several soft-set operations are examined. We establish that the soft binary piecewise plus operation is a right-left system and, under some assumptions, a commutative semigroup. Furthermore, by taking into account the algebraic properties of the operation and its distribution rules together, we demonstrate that the collection of soft sets over the universe, together with the soft binary piecewise plus operation and some other types of soft sets, form many important algebraic structures, like semirings and near semirings.

**Keywords:** Soft sets, Soft set operations, Conditional complements, Soft binary piecewise plus operation.

## 1 | Introduction

A few theories that may be used to describe uncertainty include probability theory, interval mathematics, and fuzzy set theory; however, each has drawbacks. Soft Set Theory is a novel method of characterizing uncertainty and applying it to the resolution of uncertainty-related problems. Molodtsov [1] gave the initial description of it in 1999. Since its inception, this concept has been effectively implemented in a number of mathematical domains. Among these fields studied are measurement theory, game theory, probability theory, Riemann integration, and Perron integration.

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Maji et al. [2] and Pei and Miao [3] did the initial research on soft set operations. Several soft set operations, such as restricted and extended soft set operations, were given by Ali et al. [4]. Sezgin and Atagün [5] established and provided features of the restricted symmetric difference of soft sets in their work on soft sets. They also discussed the foundations of soft-set operations and provided instances of how they relate to one another. Ali et al. [6] comprehensively analyzed the algebraic structures of soft sets. Soft set operations piqued the interest of several scholars, who conducted extensive research on the topic in [7]–[18],

The concept of the soft binary piecewise difference operation in soft sets was first proposed by Eren and Çalışıcı [19]. Additionally, a comprehensive examination of the soft binary piecewise difference operation was performed by Sezgin and Çalışıcı [20]. The extended symmetric difference of soft sets was defined, and its attributes were studied by Stojanović [21], whereas the extended difference of soft sets was first proposed by Sezgin et al. [22].

Çağman [23] presented two novel complement operations to the literature, and Sezgin et al. [24] worked on these and several other novel binary set operations. Using these new binary operations, Aybek [25] constructed a large variety of additional restricted and extended soft set operations. In their ongoing efforts to alter the structure of extended operations in soft sets, Akbulut [26], Demirci [27], and Sarıalioğlu [28] concentrated on the complementary extended soft set operations. By significantly changing the form of the soft binary piecewise operation in soft sets, [25]–[33] also looked at complementary soft binary piecewise operations. There are two important studies on soft binary piecewise operations: 1) Yavuz [38], and 2) Sezgin and Yavuz [39]. Studies [40–47] about various forms of soft equity are also essential [35]–[42].

Mathematicians have always been interested in algebraic structures, often known as mathematical systems or structures. One of the key challenges in algebraic mathematics is sorting algebraic structures based on the characteristics of the operation applied to a set. The extension of rings, including near-rings, semirings, and semifields, is one of the most well-known concepts in binary algebraic structures. For a long time, academics have been keen to understand this subject. The term semirings was initially defined by Vandiver [48]. More recently, semirings have been the subject of much research, especially their applications (see [48]). Semirings play a significant role in geometry, pure mathematics, and resolving several issues in applied mathematics and the information sciences [44]–[56].

In conclusion, semirings play a significant role in both geometry and pure mathematics. Hoorn and Rootselaar [62] discussed nearsemiring. A seminearring, often called nearsemiring in mathematics, is an algebraic structure that is more general than a near-ring or semiring. It is easy to find nearsemirings from functions on monoids. Similar to how operations from classical algebra are important to classical set theory, concepts of soft set operations are fundamental to soft sets. Therefore, if we think about the algebraic structure of soft sets in terms of, we might be able to comprehend it better.

By presenting the soft binary piecewise plus operation and closely analyzing the algebraic structures connected to it, along with other soft set operations in the collection of soft sets throughout the universe, we hope to advance the subject of soft set theory significantly. This study is organized as follows. Section 2 revisits the basic ideas of soft sets and other algebraic structures. A detailed analysis of the algebraic properties of the recently suggested soft set operation is presented in the third part. These properties allow us to prove that the soft binary piecewise plus operation is a commutative semigroup and a right-left system with the right identity empty soft set under certain conditions. The distribution of the soft binary piecewise plus operation over various soft set operations, including restricted, extended, and soft binary piecewise operations, is examined in Section 4. A detailed analysis of the algebraic structures generated by the set of soft sets with these operations is offered, considering the distribution laws and the algebraic properties of the soft set operations. It is shown that utilizing the soft binary piecewise plus operation and other types of soft sets, in the collection of soft sets over the universe, a number of important algebraic structures, including semirings and nearsemirings, may be formed. The importance of the study's findings and their potential relevance to the topic are covered in Section 5.

## 2 | Preliminaries

This section provides several algebraic structures as well as a number of basic concepts in soft set theory.

**Definition 1.** Let U be the universal set, E be the parameter set, P(U) be the power set of U, and let K ⊆ E. A pair (F, K) is called a soft set on U. Here, F is a function given by $F: K \to P(U)$ [1].

The set of all soft sets over U is denoted by $S_E$(U). Let K be a fixed subset of E, then the set of all soft sets over U with the fixed parameter set K is denoted by $S_K$(U). In other words, in the collection $S_K$(U), only soft sets with the parameter set K are included, while in the collection $S_E$(U), soft sets over U with any parameter set can be included. Clearly, the set $S_K$(U) is a subset of the set $S_E$(U).

**Definition 2.** Let (F,K) be a soft set over U. If F(e)=∅ for all e∈K, then the soft set (F,K) is called a null soft set with respect to K, denoted by $\emptyset_K$. Similarly, let (F,E) be a soft set over U. If F(e)=∅ for all e∈E, then the soft set (F,E) is called a null soft set with respect to E, denoted by $\emptyset_E$ [4].

It is known that a function F: ∅ ⟶ K, where the domain is the empty set, is referred to as the empty function. Since the soft set is also a function, it is evident that by taking the domain as ∅, a soft set can be defined as F: ∅ ⟶ P(U), where U is a universal set. Such a soft set is called an empty soft set and is denoted as $\emptyset_\emptyset$. Thus, $\emptyset_\emptyset$ is the only soft set with an empty parameter set [6].

**Definition 3.** Let (F,K) be a soft set over U. If F(e)=U for all e∈K, then the soft set (F,K) is called an absolute soft set with respect to K, denoted by $U_K$. Similarly, let (F,E) be a soft set over U. If F(e)=U for all e∈E, then the soft set (F,E) is called an absolute soft set with respect to E, denoted by $U_E$ [4].

**Definition 4.** Let (F,K) and (G,Y) be soft sets over U. If K⊆Y, and for all e∈K, F(e) ⊆G(e), then (F,K) is said to be a soft subset of (G,Y), denoted by (F,K)$\tilde{\subseteq}$(G,Y). If (G,Y) is a soft subset of (F,K), then (F,K) is said to be a soft superset of (G,Y), denoted by (F,K)$\tilde{\supseteq}$(G,Y). If (F,K)$\tilde{\subseteq}$(G,Y) and (G,Y)$\tilde{\subseteq}$(F,K), then (F,K) and (G,Y) are called soft equal sets [3].

**Definition 5.** Let (F,K) be a soft set over U. The soft complement of (F,K), denoted by $(F,K)^r = (F^r,K)$, is defined as follows: $F^r(e)=U-F(e)$, for all e∈K [4].

Çağman [23] introduced two new complements as a novel concept in set theory. For ease of representation, we denote these binary operations as + and θ, respectively. For two sets T and Y, these binary operations are defined as T+Y=T'∪Y and TθY=T'∩Y'. Sezgin et al. [24] investigated the relationship between these two operations and introduced three new binary operations, examining their relationships. For two sets T and Y, these new operations are defined as T*Y=K'∪Y', T$\gamma$Y= T'∩Y, T$\boldsymbol{\lambda}$Y=T∪Y' [24].

As a summary for soft set operations, we can categorize all types of soft set operations as follows: Let ⊗ be used to represent the set operations (i.e., here, ⊗ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then all types of soft set operations are defined as follows:

**Definition 6.** Let (F, K) and (G, Y) be two soft sets over U. The restricted ⊗ operation of (F, K) and (G, Y) is the soft set (H, P), denoted by (F, K) $\otimes_{\mathfrak{R}}$ (G, Y)= (H, P), where P = K ∩ Y≠ ∅ and for all e ∈ P, H(e) = F(e)⊗G(e). Here, if P = K ∩ Y = ∅, then (F, K) $\otimes_{\mathfrak{R}}$(G, Y)= $\emptyset_\emptyset$ [4]–[6], [25].

**Definition 7.** Let (F, K) and (G, Y) be two soft sets over U. The extended ⊗ operation (F, K) and (G,Y) is the soft set (H,P), denoted by (F, K) $\otimes_\varepsilon$(G, Y) = (H, P), where P = K ∪ Y, and for all e ∈ P [2], [4], [21], [25].

$$H(e) = \begin{cases} F(e). & e \in K - Y. \\ G(e). & e \in Y - K. \\ F(e) \otimes G(e). & e \in K \cap Y, \end{cases}$$

**Definition 8.** Let (F, K) and (G, Y) be two soft sets over U. The complementary extended ⊗ operation (F, K) and (G,Y) is the soft set (H,P), denoted by (F, K) $\overset{*}{\otimes}_{\varepsilon}$(G, Y) = (H, P), where P = K ∪ Y, and for all e ∈ P [26]–[28].

$$H(e)=\begin{cases}F'^{(e)}. & e\in K-Y.\\ G'^{(e)}. & e\in Y-K.\\ F(e)\otimes G(e). & e\in K\cap Y,\end{cases}$$

**Definition 9.** Let (F,K) and (G,Y) be two soft sets on U. The complementary soft binary piecewise ⊗ operation of (F,K) and (G,Y) is the soft set (H,K), denoted by $(F.K)\underset{\otimes}{\overset{*}{\sim}}(G.Y)=(H.K)$, where for all e ∈ K [27]–[29].

$$H(e)=\begin{cases}F'^{(e)}. & e\in K-Y.\\ F(e)\otimes G(e). & e\in K\cap Y,\end{cases}$$

**Definition 10.** Let (F,K) and (G,Y) be two soft sets on U. The soft binary piecewise ⊗ operation of (F,K) and (G,Y) is the soft set (H,K), denoted by $(F.K)\underset{\otimes}{\sim}(G.Y)=(H.K)$, where for all e ∈ K [19], [53]

$$H(e)=\begin{cases}F(e). & e\in K-Y.\\ F(e)\otimes G(e). & e\in K\cap Y,\end{cases}$$

For more about soft sets, we refer to [54]–[63].

**Definition 11.** Let (S, ⋆) be an algebraic structure. An element s ∈S is called idempotent if $s^2=s$. If $s^2=s$ for all s∈S, then the algebraic structure (S,⋆) is said to be idempotent. An idempotent semigroup is called a band, an idempotent and commutative semigroup is called a semilattice, and an idempotent and commutative monoid is called a bounded semilattice [64].

In a monoid, although the identity element is unique, a semigroup/groupoid can have one or more left identities; however, if it has more than one left identity, it does not have a right identity element; thus, it does not have an identity element. Similarly, a semigroup/groupoid can have one or more right identities; however, if it has more than one right identity, it does not have a left identity element, thus not an identity element [65].

Similarly, in a group, although each element has a unique inverse, in a monoid, an element can have one or more left inverses; however, if an element has more than one left inverse, it does not have a right inverse. Thus, it does not have an inverse. Similarly, in a monoid, an element can have one or more right inverses; however, if an element has more than one right inverse, it does not have a left inverse, thus not an inverse [65].

**Definition 12.** If a semigroup (S,*) has a left identity and every element has a right inverse, then the semigroup is called a left-right system, and if the semigroup has a right identity and every element has a left inverse, then the semigroup is called a right-left system. The difference between a left-right system and the group is that a group has a left (resp., right) identity, and every element has a left (resp., right) inverse [66].

**Definition 13.** Let S be a non-empty set, and let + and ⋆ be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a semiring:

I. (S, +) is a semigroup.

II. (S, ⋆) is a semigroup.

III. For all x, y, z ∈S, x⋆(y + z) = x⋆y + x⋆z and (x +y) ⋆z = x⋆z + y⋆z.

If for all x,y∈S, x+y=y+z, then S is called an additive commutative semiring. If x⋆y=y⋆x for all x,y∈S, then S is called a multiplicative commutative semiring. If there exists an element 1∈S such that x⋆1=1⋆x=x for all x∈S (multiplicative identity), then S is called semiring with unity. If there exists 0∈S such that for all x∈S, 0⋆x=x⋆0=0 and 0+x=x+0=x, then 0 is called the zero of S. A semiring with commutative addition and a zero element is called a hemiring [38].

**Definition 14.** Let S be a non-empty set, and let + and ⋆ be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a nearsemiring (or seminearring):

I. (S,+) is a semigroup.

II. (S, ⋆) is a semigroup.

III. For all x,y,z∈S, (x+y) ⋆z = x⋆z+y∗z (right distributivity).

If the additive zero element 0 of S (that is, for all x∈S, 0+x=0+x=x) satisfies that for all x∈S, 0⋆x=0 (left absorbing element), then (S, +, ⋆) is called a (right) nearsemiring with zero. If (S, +, ⋆) additionally satisfies x⋆0=0 for all x∈S (right absorbing element), then it is called a zero symmetric nearsemiring [47]. We refer to [67] for the possible implications of network analysis and graph applications with regard to soft sets, which are defined by the divisibility of determinants.

# 3 | Soft Binary Piecewise Plus Operation

This section presents a new soft set operation known as the soft binary piecewise plus operation. Along with presenting an example of the operation, it also examines the algebraic structures and distribution rules that the operation forms in $S_E(U)$ and the operation's overall algebraic properties and connections to other soft-set operations.

**Definition 15.** Let $(F.K)$ and $(G.Y)$ be soft sets over $U$, The soft binary piecewise plus operation of $(F.K)$ and $(G.Y)$ is the soft set $(H.K)$. denoted by, $(F.K)\ \tilde{+}\ (G.Y) = (H.K)$. where for all k∈K

$$H(k) = \begin{cases} F(k). & k \in K - Y. \\ F'(k) \cup G(k). & k \in K \cap Y, \end{cases}$$

**Example 1.** Let E=$\{e_1,e_2,e_3,e_4\}$ be the parameter set K=$\{e_1, e_4\}$ and Y=$\{e_2, e_3, e_4\}$ be the subsets of E and U=$\{h_1,h_2,h_3,h_4,h_5. h_6\}$ be the initial universe set. Assume that (F,K) and (G,Y) are the soft sets over U defined as follows:

I. (F,K)=$\{(e_{1.}\ \{h_2. h_4,h_6), (e_4,\{h_1,h_6\})\}$.

II. (G,Y)=$\{(e_2,\{h_1. h_2\}), (e_3.\{h_2,h_3,h_4. h_5\}),(e_4,\{h_2,h_4\})$.

Let (F,K) $\tilde{+}$ (G,Y)=(H,K). Then

$$H(t) = \begin{cases} F(t). & t \in K - Y. \\ F'(t) \cup G(t). & t \in K \cap Y, \end{cases}$$

Since K=$\{e_1, e_4\}$, K-Y=$\{e_1\}$ and K∩Y=$\{e_4\}$ so $H(e_1)$ =F($e_1$)=$\{h_2,h_4,h_6\}$, H($e_4$)=F'($e_4$)∪G($e_4$)= $\{h_2,h_3,h_4,h_5\} \cup \{h_2. h_4\}$. Thus(F,K) $\tilde{+}$ (G,Y)=$\{(e_1,\{h_2. h_4,h_6\}),(e_4,\{h_2. h_3. h_4. h_5\})\}$

**Theorem 1.** Algebraic properties of the operation: the set $S_E(U)$ is closed under $\tilde{+}$. That is, when (F,K) and (G,Y) are two soft sets over U, then so is (F,K) $\tilde{+}$ (G,Y).

Proof: it is clear that $\tilde{+}$ is a binary operation in $S_E(U)$. That is

$$\begin{aligned} &\tilde{+} : S_E(U) x\ S_E(U) \to S_E(U), \\ &((F,K), (G,Y)) \to (F.K)\ \tilde{+}\ (G.Y) = (H.K). \end{aligned} \quad (1)$$

Hence, the set $S_E(U)$ is closed under $\tilde{+}$. Similarly

$$\tilde{+} : S_K(U) x\ S_K(U) \to S_K(U),$$

$((F,K),(G,K)) \rightarrow (F.K) \overset{\sim}{+} (G.K) = (H.K).$

That is, when K be a fixed subset of the set E and (F,K) and (G,K) are elements of $S_K(U)$, then so is $(F,K) \overset{\sim}{+} (G,K)$. Namely, $S_K(U)$ is closed under $\overset{\sim}{+}$ either.

If $K \cap Y' \cap D = K \cap Y \cap D = \emptyset$. then $[(F.K) \overset{\sim}{+} (G,Y)] \overset{\sim}{+} (H,D) = (F,K) \overset{\sim}{+} [(G.Y) \overset{\sim}{+} (H,D)]$.

Proof: first, let's handle the Left-Hand Side (LHS) of the equality and let $(F.K) \overset{\sim}{+} (G,Y) = (T,K)$, where for all $\mathfrak{s} \in K$,

$$T(\mathfrak{s}) = \begin{cases} F(\mathfrak{s}). & \mathfrak{s} \in K - Y. \\ F'(\mathfrak{s}) \cup G(\mathfrak{s}). & \mathfrak{s} \in K \cap Y, \end{cases} \tag{2}$$

Let $(T,K) \overset{\sim}{+} (H,D) = (M,K)$, where for all $\mathfrak{s} \in K$,

$$M(\mathfrak{s}) = \begin{cases} T(\mathfrak{s}). & \mathfrak{s} \in K - D. \\ T'(\mathfrak{s}) \cup H(\mathfrak{s}). & \mathfrak{s} \in K \cap D, \end{cases}$$

Thus,

$$M(\mathfrak{s}) = \begin{cases} F(\mathfrak{s}), & \mathfrak{s} \in (K-Y)-D = K \cap Y' \cap D', \\ F'(\mathfrak{s}) \cup G(\mathfrak{s}), & \mathfrak{s} \in (K \cap Y)-D = K \cap Y \cap D', \\ F'(\mathfrak{s}) \cup H(\mathfrak{s}), & \mathfrak{s} \in (K-Y) \cap D = K \cap Y' \cap D, \\ [F(\mathfrak{s}) \cap G'(\mathfrak{s})] \cup H(\mathfrak{s}), & \mathfrak{s} \in (K \cap Y) \cap D = K \cap Y \cap D. \end{cases}$$

Now let's handle the Right-Hand Side (RHS) of the equality. Let $(G,Y) \overset{\sim}{+} (H,D) = (K,Y)$, where for all $\mathfrak{s} \in Y$

$$K(\mathfrak{s}) = \begin{cases} G(\mathfrak{s}), & \mathfrak{s} \in Y-D, \\ G'(\mathfrak{s}) \cup H(\mathfrak{s}), & \mathfrak{s} \in Y \cap D. \end{cases}$$

Let $(F.K) \overset{\sim}{+} (K,Y) = (S,K)$, where for all $\mathfrak{s} \in K$

$$S(\mathfrak{s}) = \begin{cases} F(\mathfrak{s}), & \mathfrak{s} \in K-Y, \\ F'(\mathfrak{s}) \cup K(\mathfrak{s}), & \mathfrak{s} \in K \cap Y. \end{cases}$$

Hence

$$S(\mathfrak{s}) = \begin{cases} F(\mathfrak{s}), & \mathfrak{s} \in K-Y, \\ F'(\mathfrak{s}) \cup G(\mathfrak{s}), & \mathfrak{s} \in K \cap (Y-D) = K \cap Y \cap D', \\ F'(\mathfrak{s}) \cup [G'(\mathfrak{s}) \cup H(\mathfrak{s})], & \mathfrak{s} \in K \cap (Y \cap D) = K \cap Y \cap D. \end{cases}$$

Considering K-Y in the S function, since $K-Y = K \cap Y'$, if $\mathfrak{s} \in Y'$, then $\mathfrak{s} \in D-Y$ or $\mathfrak{s} \in (Y \cup D)'$. From here, if $\mathfrak{s} \in K-Y$, then $\mathfrak{s} \in K \cap Y' \cap D'$ or $\mathfrak{s} \in K \cap Y' \cap D$. Thus, M=S for $K \cap Y' \cap D = K \cap Y \cap D = \emptyset$. That is, under suitable conditions, $\overset{\sim}{+}$ is associative in $S_E(U)$.

$$[(F.K) \overset{\sim}{+} (G,K)] \overset{\sim}{+} (H,K) \neq (F.K) \overset{\sim}{+} [(G.K) \overset{\sim}{+} (H,K)]. \tag{3}$$

Proof: let's first handle the LHS of equality. Let $(F.K) \overset{\sim}{+} (G,K) = (T,K)$, where for all $\mathfrak{s} \in K$

$$T(\mathfrak{s}) = \begin{cases} F(\mathfrak{s}), & \mathfrak{s} \in K-K = \emptyset, \\ F'(\mathfrak{s}) \cup G(\mathfrak{s}), & \mathfrak{s} \in K \cap K = K. \end{cases}$$

Let $(T,K) \overset{\sim}{+} (H,K) = (M,K)$, where for all $\mathfrak{s} \in K$

$$M(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ T'(\vartheta)\cup H(\vartheta), & \vartheta\in K\cap K=K. \end{cases}$$

Thus

$$M(\vartheta)=\begin{cases} T(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ [F(\vartheta)\cap G'(\vartheta)]\cup H(\vartheta), & \vartheta\in K\cap K=K. \end{cases}$$

Now, let's handle the RHS of equality. Let $(G,K)\tilde{+}(H,K)=(L,K)$, where for all $\vartheta\in K$

$$L(\vartheta)=\begin{cases} G(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ G'(\vartheta)\cup H(\vartheta), & \vartheta\in K\cap K=K. \end{cases}$$

Let $(F.K)\tilde{+}(L,K)=(N,K)$, where for all $\vartheta\in K$

$$N(\vartheta)=\begin{cases} F(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ F'(\vartheta)\cup L(\vartheta), & \vartheta\in K\cap K=K. \end{cases}$$

Hence

$$N(\vartheta)=\begin{cases} F(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ F'(\vartheta)\cup[G'(\vartheta)\cup H(\vartheta)], & \vartheta\in K\cap K=K. \end{cases}$$

It is seen that $M\neq N$. That is, for soft sets with the same parameter sets, $\tilde{+}$ does not have an associative property in $S_K(U)$.

$$(F.K)\tilde{+}(G,Y)\neq(G.Y)\tilde{+}(F,K). \qquad \textbf{(4)}$$

Proof: let $(F,K)\tilde{+}(G,Y)=(H,K)$, where for all $\vartheta\in K$

$$H(\vartheta)=\begin{cases} F(\vartheta), & \vartheta\in K\text{-}Y, \\ F'(\vartheta)\cup G(\vartheta), & \vartheta\in K\cap Y. \end{cases}$$

Let $(G.Y)\tilde{+}(F,K)=(T,Y)$, where for all $\vartheta\in Y$

$$T(\vartheta)=\begin{cases} G(\vartheta), & \vartheta\in Y\text{-}K, \\ G'(\vartheta)\cup F(\vartheta), & \vartheta\in Y\cap K. \end{cases}$$

Here, while the parameter set of the soft set of the LHS is K, the parameter set of the soft set of the RHS is Y. Thus, by the definition of soft equality $(F.K)\tilde{+}(G,Y)\neq(G.Y)\tilde{+}(F,K)$.

But it is obvious that $(F.K)\tilde{+}(G,K)\neq(G,K)\tilde{+}(F,K)$. So, for the fixed parameter set $K\subseteq E$, the $\tilde{+}$ operation in the $S_K(U)$ set does not have the commutative property.

$$(F.K)\tilde{+}(F,K)=U_K, \qquad \textbf{(5)}$$

Proof: let $(F.K)\tilde{+}(F,K)=(H,K)$, where for all $\vartheta\in K$

$$H(\vartheta)=\begin{cases} F(\vartheta), & \vartheta\in K\text{-}K=\emptyset, \\ F'(\vartheta)\cup F(\vartheta), & \vartheta\in K\cap K=K. \end{cases}$$

Hence, for all $\varkappa \in K$, $H(\varkappa)=F'(\varkappa)\cup F(\varkappa)=U$, thus $(H,K)=U_K$, That is, $\tilde{+}$ is not idempotent in the set $S_E(U)$.

**Theorem 2.** By *Theorem 1* and *Eqs. (1)-(5)*, when (F,K), (G,Y), and (H, D) are the elements of the set $S_E(U)$ under the condition $K\cap Y'\cap D=K\cap Y\cap D=\emptyset$, $(S_E(U),\tilde{+})$ is a noncommutative and not idempotent semigroup.

By *Theorem 1* and *Eq. (3)*, since $\tilde{+}$ is not associative in $S_K(U)$ where $K\subseteq E$ is a fixed parameter set $(S_K(U),\tilde{+})$ is not a semigroup; however, it is obviously a noncommutative groupoid.

$$(F,K)\,\tilde{+}\,\emptyset_K=(F,K)^r, \tag{6}$$

Proof: let $\emptyset_K=(S,K)$, where for all $\varkappa\in K$, $S(\varkappa)=\emptyset$, Let $(F,K)\,\tilde{+}\,(S,K)=(H,K)$, where for all $\varkappa\in K$

$$H(\varkappa)=\begin{cases} F(\varkappa), & \varkappa\in K-K=\emptyset, \\ F'(\varkappa)\cup F(\varkappa), & \varkappa\in K\cap K=K. \end{cases}$$

Thus, for all $\varkappa\in K$, $H(\varkappa)=F'(\varkappa)\cup S(\varkappa)=F'(\varkappa)\cup\emptyset=F'(\varkappa)$. Hence $(H,K)=(F,K)^r$,

$$\emptyset_K\,\tilde{+}\,(F,K)=U_K, \tag{7}$$

Proof: let $\emptyset_K=(S,K)$, where for all $\varkappa\in K$, $S(\varkappa)=\emptyset$. Let $(S,K)\,\tilde{+}\,(F,K)=(H,K)$, where for all $\varkappa\in K$

$$H(\varkappa)=\begin{cases} S(\varkappa), & \varkappa\in K-K=\emptyset, \\ S'(\varkappa)\cup F(\varkappa), & \varkappa\in K\cap K=K. \end{cases}$$

Hence, for all $\varkappa\in K$, $H(\varkappa)=S'(\varkappa)\cup F(\varkappa)=U\cup F(\varkappa)=U$ so $(H,K)=U_K$,

$$(F,K)\,\tilde{+}\,\emptyset_E=(F,K)^r. \tag{8}$$

Proof: let $\emptyset_E=(S,E)$, where for all $\varkappa\in E$, $S(\varkappa)=\emptyset$. Let $(F,K)\,\tilde{+}\,(S,E)=(H,K)$. Thus, for all $\varkappa\in K$

$$H(\varkappa)=\begin{cases} F(\varkappa), & \varkappa\in K-E=\emptyset, \\ F'(\varkappa)\cup S(\varkappa), & \varkappa\in K\cap E=K. \end{cases}$$

Hence, for all $\varkappa\in K$, $H(\varkappa)=F'(\varkappa)\cup S(\varkappa)=F'(\varkappa)\cup\emptyset=F'(\varkappa)$. Thus, $(H,K)=(F,K)^r$.

$$(F,K)\,\tilde{+}\,\emptyset_\emptyset=(F,K). \tag{9}$$

Proof: let $\emptyset_\emptyset=(S,\emptyset)$ and $(F,K)\,\tilde{+}\,(S,\emptyset)=(H,K)$, where for all $\varkappa\in K$

$$H(\varkappa)=\begin{cases} F(\varkappa), & \varkappa\in K-\emptyset=\emptyset, \\ F'(\varkappa)\cup S(\varkappa), & \varkappa\in K\cap\emptyset=\emptyset. \end{cases}$$

Hence, for all $\varkappa\in K$, $H(\varkappa)=F(\varkappa)$ so $(H,K)=(F,K)$. The right unit element of $\tilde{+}$ in the set $S_E(U)$ is the soft set $\emptyset_\emptyset$.

$$\emptyset_\emptyset\,\tilde{+}\,(F,K)=\emptyset_\emptyset. \tag{10}$$

Proof: let $\emptyset_\emptyset=(S,\emptyset)$ and $(S,\emptyset)\,\tilde{+}\,(F,K)=(H,\emptyset)$. Since $\emptyset_\emptyset$ is the only soft set whose parameter set is the empty set, $(H,\emptyset)=\emptyset_\emptyset$. That is, in $S_E(U)$, for the operation $\tilde{+}$, the left inverse of each element with respect to the right identity element $\emptyset_\emptyset$ is the soft set $\emptyset_\emptyset$, Moreover, in $S_E(U)$, the left absorbing element of the $\tilde{+}$ operation is the soft set $\emptyset_\emptyset$,

**Theorem 3.** From the properties of *Eqs. (1), (2), (9),* and *(10)*, $(S_E(U), \tilde{+})$ is a right-left system with the right identity $\emptyset_\emptyset$ and the left inverses of each element is $\emptyset_\emptyset$ under the condition $K \cap Y \cap D = K \cap Y' \cap D = \emptyset$, where (F,K), (G,Y), and (H, D) are the elements of $S_E(U)$.

$$U_K \tilde{+} (F.K) = (F.K). \tag{11}$$

Proof: let $U_K$=(T,K), where for all $\varkappa \in K$, $T(\varkappa)=U$. Let $(T,K) \tilde{+} (F.K)=(H,K)$, where for all $\varkappa \in K$

$$H(\varkappa)=\begin{cases} T(\varkappa), & \varkappa \in K-K=\emptyset, \\ T'(\varkappa) \cup F(\varkappa), & \varkappa \in K \cap K = K. \end{cases}$$

Hence for all $\varkappa \in K$, $H(\varkappa)= T'(\varkappa) \cup F(\varkappa)=\emptyset \cup F(\varkappa)=F(\varkappa)$ so $(H,K)= (F.K)$,

The left identity element of the operation $\tilde{+}$ in the set $S_K(U)$ is $U_K$.

$$(F.K) \tilde{+} U_K = U_K, \tag{12}$$

Proof: let $U_K$=(T,K), where for all $\varkappa \in K$, $T(\varkappa)=U$. Let $(F.K) \tilde{+} (T.K)=(H,K)$, where for all $\varkappa \in K$

$$H(\varkappa)=\begin{cases} F(\varkappa), & \varkappa \in K-K=\emptyset, \\ F'(\varkappa) \cup T(\varkappa), & \varkappa \in K \cap K = K. \end{cases}$$

Hence, for all $\varkappa \in K$, $H(\varkappa)=F'(\varkappa) \cup T(\varkappa)=F'(\varkappa) \cup U=U$ and so $(H,K)= U_K$,

That is, for the operation $\tilde{+}$ in $S_K(U)$ the right inverse of each element with respect to the left unit element $U_K$ is the soft set $U_K$. Moreover, the right absorbing element of the operation $\tilde{+}$ on the set $S_K(U)$ is the soft set $U_K$ ,

$$(F.K) \tilde{+} U_E = U_K, \tag{13}$$

Proof: let $U_E$=(T,E), where for all $\varkappa \in E$, $T(\varkappa)=U$. Let $(F.K) \tilde{+} (T.E)=(H,K)$, where for all $\varkappa \in K$

$$H(\varkappa)=\begin{cases} F(\varkappa), & \varkappa \in K-E=\emptyset, \\ F'(\varkappa) \cup T(\varkappa), & \varkappa \in K \cap E = K. \end{cases}$$

Hence, for all $\varkappa \in K$, $H(\varkappa)=F'(\varkappa) \cup T(\varkappa)=F'(\varkappa) \cup U=U$. Thus, $(H,K)= U_K$.

$$(F.K) \tilde{+} (F.K)^r=(F.K)^r, \tag{14}$$

Proof: let $(F.K)^r$=(H,K), where for all $\varkappa \in K$, $H(\varkappa)=F'(\varkappa)$. Let $(F.K) \tilde{+} (H.K)=(T,K)$, where for all $\varkappa \in K$

$$T(\varkappa)=\begin{cases} F(\varkappa), & \varkappa \in K-K=\emptyset, \\ F'(\varkappa) \cup H(\varkappa), & \varkappa \in K \cap K = K. \end{cases}$$

Hence, for all $\varkappa \in K$, $T(\varkappa)=F'(\varkappa) \cup H(\varkappa)=F'(\varkappa) \cup F'(\varkappa)= F'(\varkappa)$. Then $(T,K)= (F.K)^r$,

In the set $S_E(U)$, the relative complement of each soft set is its right absorbing element for the operation $\tilde{+}$.

$$(F.K)r \tilde{+} (F.K) = (F.K), \tag{15}$$

Proof: let $(F,K)^r$=(H,K), where for all $\varkappa \in K$, $H(\varkappa)=F'(\varkappa)$. Let $(H.K) \tilde{+} (F.K)=(T,K)$, where for all $\varkappa \in K$

$$T(\varkappa)=\begin{cases} H(\varkappa), & \varkappa \in K-K=\emptyset, \\ H'(\varkappa) \cup F(\varkappa), & \varkappa \in K \cap K = K. \end{cases}$$

Thus, for all $\vartheta \in K$, $T(\vartheta)= H'(\vartheta) \cup F(\vartheta)= F(\vartheta) \cup F(\vartheta)=F(\vartheta)$ so $(T,K)=(F,K)$.

In $S_E(U)$, the relative complement of each soft set is its left unit element for the operation $\widetilde{+}$.

$$[(F,K) \widetilde{+} (G,Y)]^r=(F,K) \overset{*}{\underset{\setminus}{\sim}} (G,Y). \tag{16}$$

Proof: let $(F,K) \widetilde{+} (G,Y)=(H,K)$, where for all $\vartheta \in K$

$$H(\vartheta)=\begin{cases} F(\vartheta), & \vartheta \in K-Y, \\ F'(\vartheta) \cup G(\vartheta), & \vartheta \in K \cap Y. \end{cases}$$

Let $(H,K)^r=(T,K)$, where for all $\vartheta \in K$

$$T(\vartheta)=\begin{cases} F'(\vartheta), & \vartheta \in K-Y, \\ F(\vartheta) \cup G'(\vartheta), & \vartheta \in K \cap Y. \end{cases}$$

Thus, $(T,K)=(F,K) \overset{*}{\underset{\setminus}{\sim}} (G,Y)$.

$$(F,K) \widetilde{+} (G,K) = \emptyset_K \Leftrightarrow (F,K) = U_K \text{ and } (G,K) = \emptyset_K, \tag{17}$$

Proof: let $(F,K) \widetilde{+} (G,K) = (T,K)$, where for all $\vartheta \in K$

$$T(\vartheta)=\begin{cases} F(\vartheta), & \vartheta \in K-K=\emptyset, \\ F'(\vartheta) \cup G(\vartheta), & \vartheta \in K \cap K=K. \end{cases}$$

$(T,K) = \emptyset_K$. so, for all $\vartheta \in K$, $T(\vartheta)=\emptyset$.

Hence, for all $\vartheta \in K$, $T(\vartheta)=F'(\vartheta) \cup G(\vartheta)=\emptyset \Leftrightarrow$ for all $\vartheta \in K$. $F'(\vartheta)=\emptyset$ and $G(\vartheta)=\emptyset \Leftrightarrow$ for all $\vartheta \in K$, $F(\vartheta)=U$ and $G(\vartheta)= \emptyset \Leftrightarrow (F,K) = U_K$ and $(G,K) = \emptyset_K$.

$$\emptyset_K \widetilde{\subseteq} (F,K) \widetilde{+} (G,Y) \text{ and } \emptyset_Y \widetilde{\subseteq} (G,Y) \widetilde{+} (F,K), \tag{18}$$

$$(F,K) \widetilde{+} (G,Y) \widetilde{\subseteq} U_K \text{ and } (G,Y) \widetilde{+} (F,K) \widetilde{\subseteq} U_Y, \tag{19}$$

$$(F,K)^r \widetilde{\subseteq} (F,K) \widetilde{+} (G,K) \text{ and } (G,K) \widetilde{\subseteq} (F,K) \widetilde{+} (G,K). \tag{20}$$

Proof: let $(F,K) \widetilde{+} (G,K)=(H,K)$, where for all $\vartheta \in K$

$$H(\vartheta)=\begin{cases} F(\vartheta), & \vartheta \in K-K=\emptyset, \\ F'(\vartheta) \cup G(\vartheta), & \vartheta \in K \cap K=K. \end{cases}$$

for all $\vartheta \in K$, $H(\vartheta)=F'(\vartheta) \subseteq F'(\omega) \cup G(\omega)$. $(F,K)^r \widetilde{\subseteq} (F,K) \widetilde{+} (G,K)$, $(G,K) \widetilde{\subseteq} (F,K) \widetilde{+} (G,K)$ can be shown similarly.

If $(F,K) \widetilde{\subseteq} (G,K)$, then $(H,Z) \widetilde{+} (F,K) \widetilde{\subseteq} (H,Z) \widetilde{+} (G,K)$ and $(G,K) \widetilde{+} (H,K) \widetilde{\subseteq} (F,K) \widetilde{+} (H,K)$. **(21)**

Proof: let $(F,K) \widetilde{\subseteq} (G,K)$, where for all $\vartheta \in K$, $F(\vartheta) \subseteq G(\vartheta)$. Let $(H,Z) \widetilde{+} (F,K) = (W,Z)$, where for all $\vartheta \in Z$

$$W(\vartheta)=\begin{cases} H(\vartheta), & \vartheta \in Z-K, \\ H'(\vartheta) \cup F(\vartheta), & \vartheta \in Z \cap K. \end{cases}$$

Let (H,Z) $\tilde{+}$(G. K)=(L,Z), where for all ꬶ ∈Z

$$L(ꬶ)=\begin{cases} H(ꬶ), & ꬶ\in Z\text{-}K, \\ H'(ꬶ)\cup G(ꬶ), & ꬶ\in Z\cap K. \end{cases}$$

Thus, for all ꬶ ∈Z-K, W(ꬶ)=H(ꬶ)⊆H(ꬶ)=L(ꬶ), for all ꬶ ∈Z∩K, W(ꬶ)=H'(ꬶ)∪F(ꬶ) ⊆ H'(ꬶ)∪G(ꬶ)=L(ꬶ) so (H,Z)$\tilde{+}$(F. K) $\tilde{\subseteq}$ (H,Z)$\tilde{+}$(G. K), Under the same conditions, for all ω∈K, since G' (ꬶ)∪H(ꬶ)⊆F' (ꬶ)∪H(ꬶ) (G,K) $\tilde{+}$(H. K) $\tilde{\subseteq}$ (F,K) $\tilde{+}$(H. K) is obvious.

$$\text{If } (H,Z)\,\tilde{+}(F.K)\ \tilde{\subseteq}\ (H,Z)\,\tilde{+}(G.K) \text{ then } (F,K)\ \tilde{\subseteq}\ (G.K) \text{ needs not be true.} \tag{22}$$

That is, the converse of *Theorem 1, Eq. (21)* is not true. Similarly, if (G,K)$\tilde{+}$(H. K) $\tilde{\subseteq}$(F,K)$\tilde{+}$(H. K). then (F,K)$\tilde{\subseteq}$ (G. K) needs not be true.

Proof: to demonstrate the converse of *Theorem 1. Eq. (21)* is not true; let's provide an example. Let E={$e_1$,$e_2$,$e_3$,$e_{4.}$$e_5$} be the parameter set, K={$e_1$,$e_3$} and Z={$e_1$,$e_3$. $e_5$} be two subsets of E, U={$h_1$,$h_2$. $h_3$,$h_4$. $h_5$} be the universal set. Let (F,K), (G,K), and (H,Z) be soft sets over U as follows:

(F,K)={( $e_{1.}$$h_2$. $h_5$),($e_3$,{$h_1$,$h_2$,$h_5$}), (G,K)={($e_{1.}$.{$h_2$}), ($e_3$,{$h_1$,$h_2$}),
(H,Z)={( $e_1$,∅),($e_3$,∅),($e_5$. {$h_2$. $h_5$})}. Let (H,Z)$\tilde{+}$(F. K) = (L,Z), then for all ꬶ ∈ Z-K={$e_5$},
L($e_5$)=H($e_5$)={$h_2$. $h_5$}. for all ꬶ ∈ Z ∩K={$e_1$. $e_3$}, L($e_1$)=H'($e_1$)∪F($e_1$)=U, L($e_3$)=H'($e_3$)∪F($e_3$)=U. Thus, (H,Z)$\tilde{+}$(F. K)={($e_1$,U),($e_3$,U),($e_5$. {$h_2$. $h_5$})}.

Let (H,Z)$\tilde{+}$(G. K) = (W,Z), where W($e_5$)=H($e_5$)={$h_2$. $h_5$}. W($e_1$)=H'($e_1$)∪G($e_1$)=U,
W($e_3$)=H'($e_3$)∪G($e_3$)=U. Thus, (H,Z)$\tilde{+}$(G. K)={($e_1$,U),($e_3$,U),($e_5$. {$h_2$. $h_5$})}.

Hence, (H,Z)$\tilde{+}$(F. K) $\tilde{\subseteq}$(H,Z)$\tilde{+}$(G. K), but it is clear that (F,K) is not a soft subset of (G. K). Similarly, by choosing (H,K)={( $e_1$,U),($e_3$,U)}, one can show that (G,K) $\tilde{+}$(H. K) $\tilde{\subseteq}$ (F,K) $\tilde{+}$(H. K). but (F,K) $\tilde{\subseteq}$ (G. K) is not true.

$$\text{If } (F,T)\tilde{\subseteq}(G.T) \text{ and } (K,T)\tilde{\subseteq}(L.T), \text{ then } (G,T)\tilde{+}(K,T)\tilde{\subseteq}(F,T)\tilde{+}(L,T) \text{ and } (L,T)\tilde{+}(F,T)\ \tilde{\subseteq}(K,T)\tilde{+}(G,T). \tag{23}$$

Proof: let (F,T)$\tilde{\subseteq}$ (G. T), where for all ꬶ ∈T, F(ꬶ)⊆ G(ω ) and for all ꬶ ∈T, G'(ꬶ)⊆ F'(ꬶ). Let (G,T)$\tilde{+}$(K,T)=(M,T), where for all ꬶ ∈T, M(ꬶ)=G'(ꬶ)∪K(ꬶ). Let (F,T)$\tilde{+}$(L,T)=(N,T), where for all ꬶ ∈T, N(ꬶ)=F'(ꬶ)∪L(ꬶ) and for all ꬶ ∈T, G'(ꬶ)⊆ F'(ꬶ). Thus, M(ꬶ)=G'(ꬶ)∪K(ꬶ)⊆F'(ꬶ)∪L(ꬶ)=N(ꬶ), hence (G,T)$\tilde{+}$(K,T) $\tilde{\subseteq}$(F,T)$\tilde{+}$(L,T). Under the same conditions, it can be similarly shown that (L,T)$\tilde{+}$(F,T) $\tilde{\subseteq}$(K,T)$\tilde{+}$(G,T).

# 4 | Distribution Rules

In this section, the distributions of soft binary piecewise plus operation over other soft set operations are examined in detail, and many interesting algebraic structures are obtained.

**Proposition 1.** Let (F,K), (G,Y), and (H,D) be soft sets over U. Then, the soft binary piecewise plus operation distributes over restricted operations as follows, under K∩ Y ∩ D = ∅.

$$[(F,K) \cup_R (G,Y)]\tilde{+}(H,D)=[(F.K)\tilde{+}(H,D)]\cup_R[(G.Y)\tilde{+}(H,D)]. \tag{24}$$

Proof: first, let's handle the LHS of the equality. Assume that $(F,K)\cup_R(G,Y)=(M,K\cap Y)$, where for all $\mathfrak{s}\in K\cap Y$, $M(\mathfrak{s})=F(\mathfrak{s})\cup G(\mathfrak{s})$. Let $(M,K\cap Y)\overset{\sim}{+}(H,D)=(N,K\cap Y)$, where for all $\mathfrak{s}\in K\cap Y$

$$N(\mathfrak{s})=\begin{cases} M(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)-D,\\ M'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)\cap D.\end{cases}$$

Hence

$$N(\mathfrak{s})=\begin{cases} F(\mathfrak{s})\cup G(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)-D=K\cap Y\cap D',\\ [F'(\mathfrak{s})\cap G'(\mathfrak{s})]\cup H(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)\cap D.\end{cases}$$

Now, let's handle the RHS of equality. Let $[(F.K)\overset{\sim}{+}(H,D)]\cup_R[(G.Y)\overset{\sim}{+}(H,D)]$.

Let $(F.K)\overset{\sim}{+}(H,D)=(V,K)$, where for all $\mathfrak{s}\in K$

$$V(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in K-D,\\ F'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap D.\end{cases}$$

Let $(G,Y)\overset{\sim}{+}(H,D)=(W,Y)$, where for all $\mathfrak{s}\in Y$

$$W(\mathfrak{s})=\begin{cases} G(\mathfrak{s}), & \mathfrak{s}\in Y-D,\\ G'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in Y\cap D.\end{cases}$$

Assume that $(V,K)\cup_R(W,Y)=(T,K\cap Y)$, where for all $\mathfrak{s}\in K\cap Y$, $T(\mathfrak{s})=V(\mathfrak{s})\cup W(\mathfrak{s})$

$$T(\mathfrak{s})=\begin{cases} F(\mathfrak{s})\cup G(\mathfrak{s}), & \mathfrak{s}\in(K-D)\cap(Y-D)=K\cap Y\cap D',\\ F(\mathfrak{s})\cup[G'(\mathfrak{s})\cup H(\mathfrak{s})], & \mathfrak{s}\in(K-D)\cap(Y\cap D)=\emptyset,\\ [F'(\mathfrak{s})\cup H(\mathfrak{s})]\cup G(\mathfrak{s}), & \mathfrak{s}\in(K\cap D)\cap(Y-D)=\emptyset,\\ [F'(\mathfrak{s})\cup H(\mathfrak{s})]\cup[G'(\mathfrak{s})\cup H(\mathfrak{s})], & \mathfrak{s}\in(K\cap D)\cap(Y\cap D)=K\cap Y\cap D.\end{cases}$$

Hence

$$T(\mathfrak{s})=\begin{cases} F(\mathfrak{s})\cup G(\mathfrak{s}), & \mathfrak{s}\in K\cap Y\cap D',\\ F'(\mathfrak{s})\cup G'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap Y\cap D.\end{cases}$$

Thus, it can be seen that N=T for $K\cap Y\cap D=\emptyset$.

$$[(F,K)\cap_R(G,Y)]\overset{\sim}{+}(H,D)=[(F.K)\overset{\sim}{+}(H,D)]\cap_R[(G.Y)\overset{\sim}{+}(H,D)]. \qquad \textbf{(25)}$$

**Corollary 1.** $(S_E(U),\cup_R,\overset{\sim}{+})$ is an additive commutative and additive idempotent (right) nearsemiring without zero and unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cup_R)$ is a commutative, idempotent monoid with identity $\emptyset_E$, that is, a bounded semilattice (hence a semigroup). By *Theorem 3*, $(S_E(U),\overset{\sim}{+})$ is a not idempotent, noncommutative semigroup under the condition $T\cap Z'\cap M=T\cap Z\cap M=\emptyset$, where (F,T), (G,Z), and (H,M) are soft sets. Besides, by *Proposition 1* and *Eq. (24)*, $\overset{\sim}{+}$ distributes over $\cup_R$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Thus, $(S_E(U),\cup_R,\overset{\sim}{+})$ is an additive commutative and additive idempotent (right) near-emiring without zero and unity under certain conditions.

**Corollary 2.** $(S_E(U),\cap_R,\overset{\sim}{+})$ is an additive commutative and additive idempotent (right) nearsemiring without zero and unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cap_R)$ is a commutative, idempotent monoid with identity $U_E$, hence a bounded semilattice (and consequently, a semigroup). By *Theorem 3*, $(S_E(U),\tilde{+})$ is a not idempotent, noncommutative semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z), and (H,M) are soft sets over U. Besides, by *Proposition 1* and *Eq. (25)*, $\tilde{+}$ distributes over $\cap_R$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Thus, $(S_E(U),\cap_R,\tilde{+})$ is an additive commutative and additive idempotent (right) nearsemiring without zero and unity under certain conditions.

**Proposition 2.** Let (F,K), (G,Y), and (H,D) be soft sets over U. Then, the distributions of the soft binary piecewise plus operation to extended operations are as follows.

LHS Distributions: the following hold, where $K\cap(Y\Delta D)=\emptyset$.

$$(F.K)\tilde{+}[(G,Y)\cup_\varepsilon(H,D)]=[(F,K)\tilde{+}(G,Y)]\cup_\varepsilon[(F,K)\tilde{+}(H.D)]. \quad \textbf{(26)}$$

Proof: first, let's handle the LHS of the equality. Assume that $(G.Y)\cup_\varepsilon(H,D)=(M,Y\cup D)$, where for all $\mathscr{s}\in Y\cup D$

$$M(\mathscr{s})=\begin{cases} G(\mathscr{s}), & \mathscr{s}\in Y\text{-}D,\\ H(\mathscr{s}), & \mathscr{s}\in D\text{-}Y,\\ G(\mathscr{s})\cup H(\mathscr{s}), & \mathscr{s}\in Y\cap D.\end{cases}$$

Let $(F.K)\tilde{+}(M,Y\cup D)=(N,K)$, where for all $\mathscr{s}\in K$

$$N(\mathscr{s})=\begin{cases} F(\mathscr{s}), & \mathscr{s}\in K\text{-}(Y\cup D),\\ F'(\mathscr{s})\cup M(\mathscr{s}), & \mathscr{s}\in K\cap(Y\cup D).\end{cases}$$

Hence

$$N(\mathscr{s})=\begin{cases} F(\mathscr{s}), & \mathscr{s}\in K\text{-}(Y\cup D)=K\cap Y'\cap D',\\ F'(\mathscr{s})\cup G(\mathscr{s}), & \mathscr{s}\in K\cap(Y\text{-}D)=K\cap Y\cap D',\\ F'(\mathscr{s})\cup H(\mathscr{s}), & \mathscr{s}\in K\cap(D\text{-}Y)=K\cap Y'\cap D,\\ F'(\mathscr{s})\cup[(G(\mathscr{s})\cup H(\mathscr{s})], & \mathscr{s}\in K\cap Y\cap D=K\cap Y\cap D.\end{cases}$$

Now, let's handle the RHS of the equality. Let $[(F,K)\tilde{+}(G,Y)]\cup_\varepsilon[(F,K)\tilde{+}(H.D)]$. $(F.K)\tilde{+}(G.Y)=(V,K)$, where for all $\mathscr{s}\in K$

$$V(\mathscr{s})=\begin{cases} F(\mathscr{s}), & \mathscr{s}\in K\text{-}Y,\\ F'(\mathscr{s})\cup M(\mathscr{s}), & \mathscr{s}\in K\cap Y.\end{cases}$$

Let $(F,K)\tilde{+}(H.D)=(W,K)$, where for all $\mathscr{s}\in K$

$$W(\mathscr{s})=\begin{cases} F(\mathscr{s}), & \mathscr{s}\in K\text{-}D,\\ F'(\mathscr{s})\cup H(\mathscr{s}), & \mathscr{s}\in K\cap D.\end{cases}$$

Assume that $(V,K)\cup_\varepsilon(W,K)=(T,K)$, where for all $\mathscr{s}\in K$

$$T(\mathscr{s})=\begin{cases} V(\mathscr{s}), & \mathscr{s}\in K\text{-}K=\emptyset,\\ W(\mathscr{s}), & \mathscr{s}\in K\text{-}K=\emptyset,\\ V(\mathscr{s})\cap W(\mathscr{s}), & \mathscr{s}\in K\cap K=K.\end{cases}$$

Hence

$$T(\aleph)=\begin{cases} F(\aleph)\cup F(\aleph), & \aleph\in(K-Y)\cap(K-D)=K\cap Y'\cap D', \\ F(\aleph)\cup[F'(\aleph)\cup H(\aleph)], & \aleph\in(K-Y)\cap(K\cap D)=K\cap Y'\cap D, \\ [F'(\aleph)\cup G(\aleph)]\cup F(\aleph), & \aleph\in(K\cap Y)\cap(K-D)=K\cap Y\cap D', \\ [F'(\aleph)\cup G(\aleph)]\cup[F'(\aleph)\cup H(\aleph)], & \aleph\in(K\cap Y)\cap(K\cap D)=K\cap Y\cap D. \end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(K-Y)\cap(K-D)=K\cap Y'\cap D', \\ U, & \aleph\in(K-Y)\cap(K\cap D)=K\cap Y'\cap D, \\ U, & \aleph\in(K\cap Y)\cap(K-D)=K\cap Y\cap D', \\ F'(\aleph)\cup G(\aleph)\cup H(\aleph), & \aleph\in(K\cap Y)\cap(K\cap D)=K\cap Y\cap D. \end{cases}$$

It is observed that N=T for $K\cap Y'\cap D=K\cap Y\cap D'=\emptyset$. It is evident that the condition $K\cap Y'\cap D=K\cap Y\cap D'=\emptyset$ is equivalent to the condition $K\cap(Y\Delta D)=\emptyset$.

$$(F,K)\,\tilde{+}\,[(G,Y)\cap_{\varepsilon}(H,D)]=[(F,K)\,\tilde{+}\,(G,Y)]\cap_{\varepsilon}[(F,K)\,\tilde{+}\,(H,D)]. \quad \textbf{(27)}$$

RHS distributions: the following hold, where $K\cap Y\cap D=\emptyset$.

$$[(F,K)\cap_{\varepsilon}(G,Y)]\,\tilde{+}\,(H,D)=[(F,K)\,\tilde{+}\,(H,D)]\cap_{\varepsilon}[(G,Y)\,\tilde{+}\,(H,D)]. \quad \textbf{(28)}$$

Proof: let's first handle the LHS of equality. Assume that $(F,K)\cap_{\varepsilon}(G,Y)=(M,K\cup Y)$, where for all $\aleph\in K\cup Y$

$$M(\aleph)=\begin{cases} F(\aleph), & \aleph\in K-Y, \\ G(\aleph), & \aleph\in Y-K, \\ F(\aleph)\cap G(\aleph), & \aleph\in K\cap Y. \end{cases}$$

Let $(M,K\cup Y)\,\tilde{+}\,(H,D)=(N,K\cup Y)$, where for all $\aleph\in K\cup Y$

$$N(\aleph)=\begin{cases} M(\aleph), & \aleph\in(K\cup Y)-D, \\ M'(\aleph)\cup H(\aleph), & \aleph\in(K\cup Y)\cap D. \end{cases}$$

Thus

$$N(\aleph)=\begin{cases} F(\aleph), & \aleph\in(K-Y)-D=K\cap Y'\cap D', \\ G(\aleph), & \aleph\in(Y-K)-D=K'\cap Y\cap D', \\ F(\aleph)\cap G(\aleph), & \aleph\in(K\cap Y)-D=K\cap Y\cap D', \\ F'(\aleph)\cup H(\aleph), & \aleph\in(K-Y)\cap D=K\cap Y'\cap D, \\ G'(\aleph)\cup H(\aleph), & \aleph\in(Y-K)\cap D=K'\cap Y\cap D, \\ [F'(\aleph)\cup G'(\aleph)]\cup H(\aleph), & \aleph\in(K\cap Y)\cap D=K\cap Y\cap D. \end{cases}$$

Now let's handle the RHS of the equality: $[(F,K)\,\tilde{+}\,(H,D)]\cap_{\varepsilon}[(G,Y)\,\tilde{+}\,(H,D)]$. Let $(F,K)\,\tilde{+}\,(H,D)=(V,K)$, then for all $\aleph\in K$

$$V(\text{ঌ})=\begin{cases} F(\text{ঌ}), & \text{ঌ}\in K-D,\\ F'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in K\cap D.\end{cases}$$

Let (G,Y) $\underset{+}{\sim}$ (H,D)=(W,Y), then for all ঌ∈Y

$$W(\text{ঌ})=\begin{cases} G(\text{ঌ}), & \text{ঌ}\in Y-D,\\ G'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in Y\cap D.\end{cases}$$

Assume that (V,K) $\cap_\varepsilon$ (W. Y)=(T, K∪Y), where for all ঌ∈K∪Y

$$T(\text{ঌ})=\begin{cases} V(\text{ঌ}), & \text{ঌ}\in K-Y,\\ W(\text{ঌ}), & \text{ঌ}\in Y-K,\\ V(\text{ঌ})\cap W(\text{ঌ}), & \text{ঌ}\in K\cap Y.\end{cases}$$

Hence

$$T(\text{ঌ})=\begin{cases} F(\text{ঌ}), & \text{ঌ}\in(K-D)-Y=K\cap Y'\cap D',\\ F'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in(K\cap D)-Y=K\cap Y'\cap D,\\ G(\text{ঌ}), & \text{ঌ}\in(Y-D)-K=K'\cap Y\cap D',\\ G'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in(Y\cap D)-K=K'\cap Y\cap D,\\ F(\text{ঌ})\cap G(\text{ঌ}), & \text{ঌ}\in(K-D)\cap(Y-D)=K\cap Y\cap D',\\ F(\text{ঌ})\cap[G'(\text{ঌ})\cup H(\text{ঌ})], & \text{ঌ}\in(K-D)\cap(Y\cap D)=\emptyset,\\ [F'(\text{ঌ})\cup H(\text{ঌ})]\cap G(\text{ঌ}), & \text{ঌ}\in(K\cap D)\cap(Y-D)=\emptyset,\\ [F'(\text{ঌ})\cup H(\text{ঌ})]\cap[G'(\text{ঌ})\cup H(\text{ঌ})], & \text{ঌ}\in(K\cap D)\cap(Y\cap D)=K\cap Y\cap D.\end{cases}$$

Thus

$$T(\text{ঌ})=\begin{cases} F(\text{ঌ}), & \text{ঌ}\in K\cap Y'\cap D',\\ F'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in K\cap Y'\cap D,\\ G(\text{ঌ}), & \text{ঌ}\in K'\cap Y\cap D',\\ G'(\text{ঌ})\cup H(\text{ঌ}), & \text{ঌ}\in K'\cap Y\cap D,\\ F(\text{ঌ})\cap G(\text{ঌ}), & \text{ঌ}\in K\cap Y\cap D',\\ [F'(\text{ঌ})\cap G'(\text{ঌ})]\cup H(\text{ঌ}), & \text{ঌ}\in K\cap Y\cap D.\end{cases}$$

$$[(F,K)\cup_\varepsilon (G,Y)]\underset{+}{\sim}(H,D)=[(F,K)\underset{+}{\sim}(H,D)]\cup_\varepsilon[(G.Y)\underset{+}{\sim}(H,D)]. \quad \textbf{(29)}$$

$$[(F,K)\setminus_\varepsilon (G,Y)]\underset{+}{\sim}(H,D)=[(F,K)\underset{+}{\sim}(H,D)]\setminus_\varepsilon[(G.Y)\underset{+}{\sim}(H,D)]. \quad \textbf{(30)}$$

$$[(F,K)\,\Delta_\varepsilon(G,Y)]\underset{+}{\sim}(H,D)=[(F,K)\underset{+}{\sim}(H,D)]\Delta_\varepsilon[(G.Y)\underset{+}{\sim}(H,D)]. \quad \textbf{(31)}$$

$$[(F,K)+_\varepsilon(G,Y)]\underset{+}{\sim}(H,D)=[(F,K)\underset{+}{\sim}(H,D)]+_\varepsilon[(G.Y)\underset{+}{\sim}(H,D)]. \quad \textbf{(32)}$$

$$[(F,K)\,\gamma_\varepsilon(G,Y)]\underset{+}{\sim}(H,D)=[(F,K)\underset{\gamma}{\sim}(H,D)]\gamma_\varepsilon[(G.Y)\underset{\gamma}{\sim}(H,D)]. \quad \textbf{(33)}$$

$$[(F,K) *_{\varepsilon}(G,Y)]\tilde{+}(H,D)=[(F,K)\tilde{+}(H,D)]*_{\varepsilon}[(G,Y)\tilde{+}(H,D)]. \quad (34)$$

$$[(F,K)\,\theta_{\varepsilon}(G,Y)]\tilde{+}(H,D)=[(F,K)\tilde{+}(H,D)]\theta_{\varepsilon}[(G,Y)\tilde{+}(H,D)]. \quad (35)$$

**Corollary 3.** $(S_E(U), \cup_{\varepsilon},\tilde{+})$ and $(S_E(U),\cap_{\varepsilon},\tilde{+})$ are additive commutative and additive idempotent (right) nearsemirings with zero but without unity and zero symmetric properties under certain conditions. Similarly, $(S_E(U),\setminus_{\varepsilon},\tilde{\theta})$, $(S_E(U),\Delta_{\varepsilon},\tilde{\theta})$, $(S_E(U),+_{\varepsilon},\tilde{\theta})$, $(S_E(U),\gamma_{\varepsilon},\tilde{+})$, $(S_E(U),\lambda_{\varepsilon},\tilde{+})$, $(S_E(U),*_{\varepsilon},\tilde{+})$, $(S_E(U),\theta_{\varepsilon},\tilde{+})$ are additive commutative not idempotent (right) nearsemirings with zero, but without unity and zero symmetric property under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U), \cup_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_{\emptyset}$, that is, a bounded semilattice (hence a semigroup). By *Theorem 3*, $(S_E(U),\tilde{+})$ is a noncommutative and not idempotent semigroup under the condition T∩Z'∩M = T∩Z∩M =∅, where (F,T), (G,Z), and (H,M) are soft sets over U. Besides, by *Theorem 2*, $\emptyset_{\emptyset}\tilde{+}(F,T)=\emptyset_{\emptyset}$, that is $\emptyset_{\emptyset}$ is the left absorbing element for $\tilde{+}$ in $S_E(U)$, furthermore, by *Proposition 2*, $\tilde{+}$ distributes over $\cup_{\varepsilon}$ from RHS under the condition T∩Z∩M=∅. Thus, $(S_E(U), \cup_{\varepsilon}.\tilde{+})$ is an additive commutative and additive idempotent (right) nearsemirings with zero but without unity under certain conditions.

Moreover, since $(F,K)\tilde{+}\emptyset_{\emptyset} \neq \emptyset_{\emptyset}$, $(S_E(U),\cup_{\varepsilon},\tilde{+})$ is a (right) nearsemiring without zero symmetric property. Similarly, $(S_E(U),\cap_{\varepsilon},\tilde{+})$ is an additive commutative and additive idempotent (right) nearsemirings with zero but without unity and zero symmetric properties under certain conditions. Furthermore, $(S_E(U),\setminus_{\varepsilon},\tilde{+})$, $(S_E(U),\Delta_{\varepsilon},\tilde{+})$, $(S_E(U),+_{\varepsilon},\tilde{+})$, $(S_E(U),\gamma_{\varepsilon},\tilde{+})$, $(S_E(U),\lambda_{\varepsilon},\tilde{+})$, $(S_E(U),*_{\varepsilon},\tilde{+})$, $(S_E(U),\theta_{\varepsilon},\tilde{+})$ are all additive commutative, not idempotent (right) nearsemirings with zero but without unity and zero symmetric property under certain conditions. Here, note that Aybek [25] showed that the first operation is associative in $S_E(U)$ under the condition T∩Z∩M=∅ (for $\Delta_{\varepsilon}$, without any condition).

**Corollary 4.** $(S_E(U), \cup_{\varepsilon},\tilde{+})$ and $(S_E(U), \cap_{\varepsilon},\tilde{+})$ are additive commutative and additive idempotent semirings without zero and without unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cup_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_{\emptyset}$, that is, a bounded semilattice (hence a semigroup). By *Theorem 3*, $(S_E(U),\tilde{+})$ is a noncommutative and not idempotent semigroup under the condition T∩Z'∩M = T∩Z∩M =∅, where (F,T), (G,Z), and (H,M) are soft sets over U. Besides, by *Proposition 2*, $\tilde{+}$ distributes over $\cup_{\varepsilon}$ from LHS under the condition T∩(ZΔM) =∅, and $\tilde{+}$ distributes over $\cup_{\varepsilon}$ from RHS under the condition T∩Z∩M=∅. Thus, $(S_E(U), \cup_{\varepsilon},\tilde{+})$ is an additive commutative and additive idempotent semiring without zero and unity under certain conditions. One can similarly show that $(S_E(U), \cap_{\varepsilon},\tilde{+})$ is an additive commutative and additive idempotent semiring without zero and unity under certain conditions.

**Proposition 3.** Let (F,K), (G,Y), and (H,D) be soft sets on U. Then, the distribution of the soft binary piecewise plus operation over soft binary piecewise operations are as follows:

The following hold, where K∩Y∩D=∅.

$$[(F,K)\,\tilde{\cap}\,(G,Y)]\tilde{+}(H,D)=[(F,K)\tilde{+}(H,D)]\,\tilde{\cap}\,[(G,Y)\tilde{+}(H,D)]. \quad (36)$$

Proof: let's first handle the LHS of equality. Assume that $(F,K)\tilde{\cap}(G,Y)=(M,K)$, where for all $\mathcal{S}\in K$

$$M(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in K\text{-}Y,\\ F(\mathfrak{s})\cap G(\mathfrak{s}), & \mathfrak{s}\in K\cap Y.\end{cases}$$

Let $(M,K)\ \tilde{+}\ (H,D)=(N,K)$, where for all $\mathfrak{s}\in K$

$$N(\mathfrak{s})=\begin{cases} M(\mathfrak{s}), & \mathfrak{s}\in K\text{-}D,\\ M'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap D.\end{cases}$$

Thus

$$N(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in (K\text{-}Y)\text{-}D=K\cap Y'\cap D',\\ F(\mathfrak{s})\cap G(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)\text{-}D=K\cap Y\cap D',\\ F'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in (K\text{-}Y)\cap D=K\cap Y'\cap D,\\ [F'(\mathfrak{s})\cup G'(\mathfrak{s})]\cup H(\mathfrak{s}), & \mathfrak{s}\in (K\cap Y)\cap D=K\cap Y\cap D.\end{cases}$$

Now, let's handle the RHS of equality. Assume that $[(F,K)\tilde{+}(H,D)]\tilde{\cap}[(G,Y)\ \tilde{+}\ (H,D)]$. $(F,K)\ \tilde{+}\ (H,D)=(V,K)$, where for all $\mathfrak{s}\in K$

$$V(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in K\text{-}D,\\ F'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap D.\end{cases}$$

Let $(G,Y)\tilde{+}\ (H,D)=(W,Y)$, where for all $\mathfrak{s}\in Y$

$$W(\mathfrak{s})=\begin{cases} G(\mathfrak{s}), & \mathfrak{s}\in Y\text{-}D,\\ G'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in Y\cap D.\end{cases}$$

Assume that $(V,K)\ \tilde{\cap}\ (W,Y)=(T,K)$, where for all $\mathfrak{s}\in K$

$$T(\mathfrak{s})=\begin{cases} V(\mathfrak{s}), & \mathfrak{s}\in K\text{-}Y,\\ V(\mathfrak{s})\cap W(\mathfrak{s}), & \mathfrak{s}\in K\cap Y.\end{cases}$$

Hence

$$T(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in (K\text{-}D)\text{-}Y=K\cap Y'\cap D',\\ F'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in (K\cap D)\text{-}Y=K\cap Y'\cap D,\\ F(\mathfrak{s})\cap G(\mathfrak{s}), & \mathfrak{s}\in (K\text{-}D)\cap (Y\text{-}D)=K\cap Y\cap D',\\ F(\mathfrak{s})\cap [G'(\mathfrak{s})\cup H(\mathfrak{s})], & \mathfrak{s}\in (K\text{-}D)\cap (Y\cap D)=\emptyset,\\ [F'(\mathfrak{s})\cup H(\mathfrak{s})]\cap G(\mathfrak{s}), & \mathfrak{s}\in (K\cap D)\cap (Y\text{-}D)=\emptyset,\\ [F'(\mathfrak{s})\cup H(\mathfrak{s})]\cap [G'(\mathfrak{s})\cup H(\mathfrak{s})], & \mathfrak{s}\in (K\cap D)\cap (Y\cap D)=K\cap Y\cap D.\end{cases}$$

Thus

$$T(\mathfrak{s})=\begin{cases} F(\mathfrak{s}), & \mathfrak{s}\in K\cap Y'\cap D',\\ F'(\mathfrak{s})\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap Y'\cap D,\\ F(\mathfrak{s})\cap G(\mathfrak{s}), & \mathfrak{s}\in K\cap Y\cap D',\\ [F'(\mathfrak{s})\cap G'(\mathfrak{s})]\cup H(\mathfrak{s}), & \mathfrak{s}\in K\cap Y\cap D. \end{cases}$$

Thus, it can be seen that N=T for $K\cap Y\cap D=\emptyset$.

$$[(F, K) \widetilde{\cup} (G,Y)]\widetilde{+}(H,D)=[(F,K)\widetilde{+}(H,D)] \widetilde{\cup} [(G. Y) \widetilde{+} (H,D)]. \quad (37)$$

$$[(F, K) \widetilde{\setminus} (G,Y)]\widetilde{+}(H,D)=[(F,K)\widetilde{+}(H,D)] \widetilde{\setminus} [(G. Y) \widetilde{+} (H,D)]. \quad (38)$$

$$[(F, K)\widetilde{\Delta}(G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{\Delta} [(G. Y) \widetilde{+} (H,D)]. \quad (39)$$

$$[(F, K) \widetilde{+} (G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{+} [(G. Y) \widetilde{+} (H,D)]. \quad (41)$$

$$[(F, K) \widetilde{\gamma} (G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{\gamma} [(G. Y) \widetilde{+} (H,D)]. \quad (42)$$

$$[(F, K) \widetilde{*} (G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{*} [(G. Y) \widetilde{+} (H,D)]. \quad (43)$$

$$[(F, K) \widetilde{\theta} (G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{\theta} [(G. Y)\widetilde{+} (H,D)]. \quad (44)$$

$$[(F,K) \widetilde{\lambda} (G,Y)]\widetilde{+} (H,D)=[(F,K) \widetilde{+}(H,D)] \widetilde{\lambda} [(G. Y) \widetilde{+}(H,D)]. \quad (45)$$

**Corollary 5.** $(S_E(U),\widetilde{\cap},\widetilde{+})$ and $(S_E(U),\widetilde{\cup},\widetilde{+})$ are additive idempotent, noncommutative (right) nearsemirings without zero and unity under certain conditions.

Proof: Yavuz [53] showed that $(S_E(U),\widetilde{\cap})$ and $(S_E(U),\widetilde{\cup})$ are idempotent, noncommutative semigroups (that is a band) under the condition $T\cap Z'\cap M=\emptyset$, where (F,T), (G,Z), and (H,M) are soft sets over U. By *Theorem 3*, $(S_E(U),\widetilde{+})$ is a noncommutative and not idempotent semigroup under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z), and (H,M) are soft sets over U. Besides, by *Proposition 3*, $\widetilde{+}$ distributes over $\widetilde{\cap}$ and $\widetilde{\cup}$ from RHS under the condition $T\cap Z\cap M=\emptyset$. Consequently, $(S_E(U),\widetilde{\cap}, \widetilde{+})$ and $(S_E(U),\widetilde{\cup}, \widetilde{+})$ are additive idempotent, noncommutative (right) nearsemirings without zero and unity under certain conditions.

**Corollary 5.** $(S_E(U),\widetilde{\setminus},\widetilde{+})$, $(S_E(U),\widetilde{\Delta},\widetilde{+})$, $(S_E(U),\widetilde{+},\widetilde{+})$, $(S_E(U),\widetilde{\gamma},\widetilde{+})$, $(S_E(U),\widetilde{*},\widetilde{+})$, $(S_E(U),\widetilde{\theta},\widetilde{+})$ are all not idempotent, and noncommutative (right) nearsemirings without zero and unity under the condition $T\cap Z'\cap M = T\cap Z\cap M=\emptyset$, where (F,T), (G,Z), and (H,M) are soft sets over U. Here note that Yavuz [53] showed that the first operation is associative in $S_E(U)$ under the condition $T\cap Z'\cap M=T\cap Z\cap M=\emptyset$ (for $\widetilde{\Delta}$, under the condition $T\cap Z'\cap M=\emptyset$).

## 5 | Conclusion

Parametric approaches like soft sets and soft operations are useful when working with uncertain data. New methods for solving parametric data issues may be gained by introducing new soft operations and determining their algebraic properties and applications. In this sense, the work introduces a special form of soft-set operation. We aim to significantly advance the field of soft set theory by proposing a new soft set operation that we term the soft binary piecewise plus operation and closely studying the algebraic structures underlying it as well as other new soft set operations in the class of soft sets.

In particular, all the algebraic features of this novel soft set operation are thoroughly examined, and the distributions of the soft binary piecewise plus operation over various types of soft set operations are explored. Considering the distribution laws and the algebraic properties of the soft set operations, an extensive analysis of the algebraic structures generated by the set of soft sets with these operations is presented. We demonstrate how various types of soft sets, in the collection of soft sets over the universe with the soft binary piecewise plus operation, form several important algebraic structures, including semirings and nearsemirings:

I. $(S_E(U),\widetilde{+})$ is a noncommutative and not idempotent semigroup under certain conditions, moreover $(S_E(U),\widetilde{+})$ is a right-left system under certain conditions.

II. $(S_E(U),\cup_R,\widetilde{+})$, $(S_E(U),\cap_R,\widetilde{+})$ are additive commutative and additive idempotent (right) nearsemirings without zero and unity under certain conditions.

III. $(S_E(U), \cup_\varepsilon,\widetilde{+})$ and $(S_E(U),\cap_\varepsilon,\widetilde{+})$ are additive commutative and additive idempotent (right) nearsemirings with zero but without unity and zero symmetric properties under certain conditions.

IV. $(S_E(U),\backslash_\varepsilon,\widetilde{+})$, $(S_E(U),\Delta_\varepsilon,\widetilde{+})$, $(S_E(U),+_\varepsilon,\widetilde{+})$, $(S_E(U),\gamma_\varepsilon,\widetilde{+})$, $(S_E(U),\lambda_\varepsilon,\widetilde{+})$, $(S_E(U),*_\varepsilon,\widetilde{+})$, $(S_E(U),\theta_\varepsilon,\widetilde{+})$ are additive commutative not idempotent (right) nearsemirings with zero, but without unity and zero symmetric property under certain conditions.

V. $(S_E(U),\cup_\varepsilon,\widetilde{+})$ and $(S_E(U),\cap_\varepsilon,\widetilde{+})$ are additive commutative and additive idempotent semirings without zero and unity under certain conditions.

VI. $(S_E(U),\widetilde{\cap},\widetilde{+})$ and $(S_E(U),\widetilde{\cup},\widetilde{+})$ are additive idempotent, noncommutative (right) nearsemirings without zero and unity under certain conditions.

VII. $(S_E(U),\widetilde{\backslash},+)$ are all noncommutative and not idempotent (right) nearsemirings without zero and unity under certain conditions.

By examining novel soft set operations and the algebraic structures of soft sets, we can comprehend their application fully. This could advance the subjects of soft set theory and classical algebraic literature in addition to providing new examples of algebraic structures. This work aims to get the specific algebraic structures formed in the collection of soft sets established over a universal set by the soft binary piecewise plus operation combined with various types of soft set operations. This type of in-depth research ought to enhance our comprehension of the use of soft sets. Other studies may thoroughly investigate more soft binary piecewise operations versions and their accompanying distributions and attributes.

# Author Contributions

All authors contributed to the study's conception and design. AS performed material preparation, data collection, and analysis. EY wrote the first draft of the manuscript. All authors read and approved the final manuscript.

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## Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author (Aslıhan Sezgin, aslihan.sezgin@amasya.edu.tr) on reasonable request.

## Conflict Of Interest

The authors stated that there are no conflicts of interest regarding the publication of this article.

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