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Some New Operations on Pythagorean Fuzzy Sets

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Abstract

The concept of Pythagorean fuzzy sets (PFSs) was initially developed by Yager in 2013, which provides a novel way to model uncertainty and vagueness with high precision and accuracy compared to Pythagorean fuzzy sets (PFSs).The concept was concretely designed to represent uncertainty and vagueness in mathematical way and to furnish a formalized tool for tackling imprecision to real problems. In this paper, various operations in Pythagorean Fuzzy Sets are discussed. Some theorems are proved for establishing the properties of Pythagorean fuzzy operators with respect to different Pythagorean fuzzy sets.

Keywords: Intuitionistic fuzzy sets, Pythagorean fuzzy sets, Operations on pythagorean fuzzy sets.

1|Introduction

Zadeh [24] introduced the idea of fuzzy set which has a membership function, μ that assigns to each element of the universe of discourse, a number from the unit interval [0, 1] to indicate the degree of belongingness to the set under consideration. The notion of fuzzy sets generalizes classical sets theory by allowing intermediate situations between the whole and nothing. In a fuzzy set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means belongs, and other values indicate the degree of membership

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to a class. For fuzzy sets, the membership function replaced the characteristic function in crisp sets. Since the pioneering work of Zadeh, the fuzzy set theory has been used in different disciplines such as management sciences, engineering, mathematics, social sciences, statistics, signal processing, artificial intelligence, automata theory, medical and life sciences.

The concept of fuzzy sets theory seems to be inconclusive because of the exclusion of nonmembership function and the disregard for the possibility of hesitation margin. Atanassov [7] critically studied these short comings and proposed a concept called intuitionistic fuzzy sets (IFSs) . The construct (that is, IFSs) incorporates both membership function, μ and nonmembership function, ν with hesitation margin, π (that is, neither membership nor nonmembership functions), such that $\mu+\nu \leq 1$ and $\mu+\nu+\pi=1$. The notion of IFSs provides a flexible framework to elaborate uncertainty and vagueness. There are lot of research work done in area of IFSs in [4, 10].

There are many situations, where $\mu+\nu \geq 1$ which violate the conditions IFSs. This limitation in IFS naturally led to a construct a new idea, called Pythagorean fuzzy sets (PFSs). Pythagorean fuzzy sets (PFSs) proposed by Yager [22], a new tool to deal with vagueness considering the membership grade (μ) and non-membership grade (ν) satisfying the conditions $0 \leq \mu \leq 1$; $0 \leq \nu \leq 1$, and also, it follows that $\mu^2+\nu^2+\pi^2 \leq 1$, where π is the hesitant index of Pythagorean fuzzy sets. PFS is more capable than IFS to model the vagueness in the practical problem.

In this paper, we delve into the realm of Pythagorean fuzzy sets and explore various operations that can be performed within this framework. These operations play a pivotal role in manipulating Pythagorean fuzzy sets, facilitating effective decision-making and inference processes in diverse applications such as expert systems, pattern recognition, decision analysis, and more. Through theoretical analysis and illustrative examples, we aim to elucidate the underlying principles and significance of these operations, showcasing their applicability and effectiveness in handling uncertainty and vagueness inherent in real-world decision scenarios.

The rest of the paper organized as follows. In Section 2, the preliminaries and some definitions are given and present some operations of Pythagorean fuzzy sets. In Section 3, introduce the new operations of Pythagorean fuzzy sets and discussed some important results Pythagorean fuzzy sets. At the end, a conclusion is made in Section 4.

2|Preliminary Concepts

This part provides a concise overview of the key concepts and outcomes that are essential for understanding the subsequent sections. In this text, we discussed fundamental concepts fuzzy sets, Intuitionistic fuzzy sets and Pythagorean fuzzy set that are used in the rest of the paper.

Definition 1. *The fuzzy set A is defined as the collection of pairs*

$$A=(\xi, \alpha_A(\xi)),$$

where ξ belongs to X, a universal set. Here, $\alpha_A(\xi)$ represents the membership function of ξ in A, which assigns a real number between 0 and 1 to each element in X.

Definition 2. *An Intuitionistic fuzzy set (IFS), denoted by A, is an entity that exists in a nonempty set X and is is defined as the collection of pairs*

$$A=(\xi, \alpha_A(\xi), \beta_A(\xi)),$$

where ξ belongs to X. The degree of membership function $\alpha_A(\xi)$, maps elements from X to the interval $[0,1]$. The non-membership function $\beta_A(\xi)$ maps the set X to the interval $[0,1]$. They satisfy the condition

$$0 \leq \alpha_A(\xi))+\beta_A(\xi) \leq 1$$

for every $\xi \in X$. An Intuitionistic fuzzy set A is represented symbolically as $A=(\alpha_A, \beta_A)$.

The degree of indeterminacy $h_A(\xi)=\sqrt{1-\alpha_A(\xi)-\beta_A(\xi)}$.

In practice, the condition $0 \leq \rho(\xi)+\sigma(\xi) \leq 1$ may not be true for any reason. For example $0.5+0.7=1.2>1$, but $0.5^2+0.7^2<1$, or $0.6+0.6=1.2>1$, but $0.6^2+0.6^2<1$. To address this issue, Yager [21, 22] proposed the provide the notion of the Pythagorean fuzzy set in 2013.

Definition 3. *A Pythagorean fuzzy set, $\hat{P}$ in a finite universe of discourse X is given by*

$$\hat{P} = \{\langle \xi, \rho_{\hat{P}}(\xi), \sigma_{\hat{P}}(\xi)\rangle | \xi \in X\},$$

where $\rho_{\hat{P}}(\xi) : X \to [0,1]$ indicates the grade to which the element $\xi \in X$ and $\sigma_{\hat{P}}(\xi) : X \to [0,1]$ represents the grade to which the element $\xi \in X$ is not a member of the set $\hat{P}$, with the condition that

$$0 \leq (\rho_{\hat{P}}(\xi))^2 + (\sigma_{\hat{P}}(\xi))^2 \leq 1,$$

for all $\xi \in X$.

The degree of indeterminacy $h_{\hat{P}}(\xi) = \sqrt{1 - (\rho_{\hat{P}}(\xi))^2 - (\sigma_{\hat{P}}(\xi))^2}$.

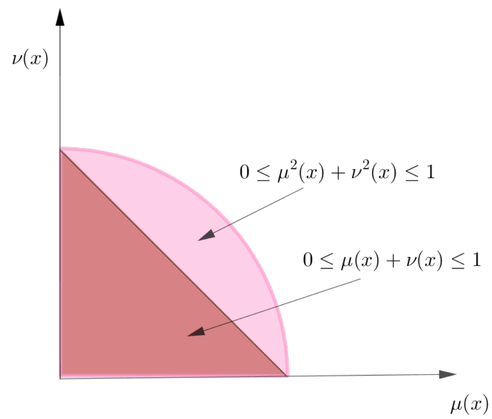

FIGURE 1. Comparison Spaces for Intuitionistic Fuzzy and Pythagorean membership grades

3.|Some Operation of Pythagorean Fuzzy sets

Let P_1 and P_2 be two Pythagorean fuzzy sets, then the following operations and relations can be defined as
$P_1 \subseteq P_2$ iff $(\rho_{P_1}(x) \leq \rho_{P_2}(x))$ and $(\sigma_{P_1}(x) \geq \sigma_{P_2}(x))$ (for all $x \in E$)
$P_1 = P_2$ iff $(\rho_{P_1}(x) = \rho_{P_2}(x))$ and $(\sigma_{P_1}(x) = \sigma_{P_2}(x))$ (for all $x \in E$)
$P_1 \cap P_2 = \{\langle x. \min(\rho_{P_1}(x), \rho_{P_2}(x)), \max(\sigma_{P_1}(x), \sigma_{PP_2}(x))\rangle : x \in E\}$
$P_1 \cup P_2 = \{\langle x. \max(\rho_{P_1}(x), \rho_{P_2}(x)), \min(\sigma_{P_1}(x), \sigma_{P_2}(x))\rangle : x \in E\}$
$p_1 + P_2 = \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x) - \rho_{P_1}(x)\rho_{P_2}(x,), \sigma_{P_1}(x).\sigma_{P_2}(x)\rangle : x \in E\}$
$P_1.P_2 = \{\langle x, \rho_{P_1}(x)\rho_{P_2}(x), \sigma_{P_1}(x) + \sigma_{P_2}(x) - \sigma_{P_1}(x).\sigma_{P_2}(x)\rangle : x \in E\}$
$P_1 @ P_2 = \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x)/2, \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\}$

Theorem 1. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1}), P_2 = (\rho_{P_2}, \sigma_{P_2})$ and $P_3 = (\rho_{P_3}, \sigma_{P_3})$ be three Pythagorean fuzzy sets. Then $(P_1 \cap P_2)@P_3 = (P_1@P_3) \cap (P_2@P_3)$.*

Proof: From definition, we have
$P_1 \cap P_2 = \{\langle x, \min \rho_{P_1}(x), \rho_{P_2}(x), \max \sigma_{P_1}(x), \sigma_{P_2}(x)\rangle : x \in E\}$
and $P_1@P_2 = \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x)/2, \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\}$
Now,

$$\begin{aligned}
(P_1 \cap P_2)@P_3 &= \{\langle x, \min\{\rho_{P_1}(x), \rho_{P_2}(x)\}, \max\{\sigma_{P_1}(x), \sigma_{P_2}(x)\}\rangle : x \in E\}, \\
&\quad @\{\langle x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\
&\quad Let \rho_{P_1}(x) < \rho_{P_2}(x) \text{ and } \sigma_{P_2}(x) > \sigma_{P_1}(x) \\
&= \{\langle x, \rho_{P_1}(x), \sigma_{P_2}(x)\rangle : x \in E\}@\{\langle x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\
&= \{\langle x, \rho_{P_1}(x) + \rho_{P_3}(x)/2, \sigma_{P_2}(x) + \sigma_{P_3}(x)/2\rangle : x \in E\} \qquad (1)
\end{aligned}$$

Again,

$$\begin{aligned}(P_1@P_3)\cap(P_2@P_3) &= \{\langle x, \rho_{P_1}(x)+\rho_{P_3}(x)/2, \sigma_{P_1}(x)+\sigma_{P_3}(x)/2\rangle : x\in E\}\\ &\quad \cap\{\langle x, \rho_{P_1}(x)+\rho_{P_3}(x)/2, \sigma_{P_2}(x)+\rho_{P_3}(x)/2\rangle : x\in E\}\\ &= \{\langle x, \min\{\rho_{P_1}(x)+\rho_{P_2}(x)+\rho_{P_3}(x)/2\},\\ &\quad \max\{(\sigma_{P_1}(x)+\sigma_{P_3}(x))/2; (\sigma_{P_2}(x)+\sigma_{P_3}(x))/2\} : x\in E\}\\ &= \{\langle x, \rho_{P_1}(x)+\rho_{P_3}(x)/2, \sigma_{P_2}(x)+\rho_{P_3}(x)/2\rangle : x\in E\} \qquad (2)\end{aligned}$$

From equation (1) and (2), we get
$(P_1\cap P_2)@P_3 = (P_1@P_3)\cap(P_2@P_3)$.

Theorem 2. *Let $P_1=(\rho_{P_1},\sigma_{P_1}), P_2=(\rho_{P_2},\sigma_{P_2})$ and $P_3=(\rho_{P_3},\sigma_{P_3})$ be three Pythagorean fuzzy sets, then* $P_1@(P_2\cap P_3) = (P_1@P_2)\cap(P_1@P_3)$

Proof: We know that, $P_1\cap P_2 = \{\langle x, \min\rho_{P_1}(x), \rho_{P_2}(x), \max\sigma_{P_1}(x), \sigma_{P_2}(x)\rangle : x\in E\}$
$P_1@P_2 = \{\langle x, \rho_{P_1}(x)+\rho_{P_2}(x)/2, \sigma_{P_1}(x)+\sigma_{P_2}(x)/2\rangle : x\in E\}$

$$\begin{aligned}P_1@(P_2\cap P_3) &= \{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x\in E\}\\ &\quad @\{\langle x, \min\{\rho_{P_2}(x), \rho_{P_3}(x)\}, \max\{\sigma_{P_2}(x), \sigma_{P_3}(x)\}\rangle : x\in E\}\\ &\quad Let \rho_{P_2}(x) < \rho_{P_3}(x) \text{ and } \sigma_{P_2}(x) < \sigma_{P_3}(x)\\ &= \{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x\in E\}@\{\langle x, \rho_{P_2}(x), \sigma_{P_3}(x)\rangle : x\in E\}\\ &= \{\langle x, \rho_{P_1}(x)+\rho_{P_2}(x)/2, \sigma_{P_1}(x)+\sigma_{P_3}(x)/2\rangle : x\in E\} \qquad (3)\end{aligned}$$

Also,

$$\begin{aligned}(P_1@P_2)\cap(P_1@P_3) &= \{\langle x, \rho_{P_1}(x)+\rho_{P_2}(x)/2, \sigma_{P_1}(x)+\sigma_{P_2}(x)/2\rangle : x\in E\}\\ &\quad \cap\{\langle x, \rho_{P_1}(x)+\rho_{P_3}(x)/2, \sigma_{P_1}(x)+\rho_{P_3}(x)/2\rangle : x\in E\}\\ &= \{\langle x, \min\{\rho_{P_1}(x)+\rho_{P_2}(x)+\rho_{P_3}(x)/2\},\\ &\quad \max\{(\sigma_{P_1}(x)+\sigma_{P_2}(x))/2; (\sigma_{P_1}(x)+\sigma_{P_3}(x))/2\}\rangle : x\in E\}\\ &= \{\langle x, \rho_{P_1}(x)+\rho_{P_2}(x)/2, \sigma_{P_1}(x)+\rho_{P_3}(x)/2\rangle : x\in E\} \qquad (4)\end{aligned}$$

From equation (3) and (4), conveys
$P_1@(P_2\cap P_3) = (P_1@P_2)\cap(P_1@P_3)$.

Theorem 3. *Let $P_1=(\rho_{P_1},\sigma_{P_1})$, $P_2=(\rho_{P_2},\sigma_{P_2})$ and $P_3=(\rho_{P_3},\sigma_{P_3})$ be three Pythagorean fuzzy sets. Then* $(P_1\cup P_2)@P_3 = (P_1@P_3)\cup(P_2@P_3)$

Proof: For three PFSs A, B and C, from definition

$$\begin{aligned}(P_1\cup P_2)@P_3 &= \{\langle x, \max\{\rho_{P_1}(x), \rho_{P_2}(x)\}, \min\{\sigma_{P_1}(x), \sigma_{P_2}(x)\rangle : x\in E\}@\{\langle x, \rho_C(x), \sigma_C x)\rangle : x\in E\}\\ &\quad Let \rho_A < \rho_B, \text{ and } \sigma_A(x) < \sigma_B(x)\\ &= \{\langle x, , \rho_B(x), \sigma_A(x)\rangle : x\in E\}@\{\langle x, \rho_C(x), \sigma_C(x)\rangle : x\in E\}\\ &= \{\langle x, \rho_B(x)+\rho_C(x)/2, \sigma_A(x)+\sigma_C(x)/2\rangle : x\in E\} \qquad (5)\end{aligned}$$

Also,

$$\begin{aligned}(P_1@P_3)\cup(P_2@P_3) &= \{\langle x, \rho_{P_1}(x)+\rho_{P_3}(x)/2, \sigma_{P_1}(x)+\sigma_{P_3}(x)/2\rangle : x\in E\}\\ &\quad \cup\{\langle x, \rho_{P_2}(x)+\rho_{P_3}(x)/2.\sigma_{P_2}(x)+\sigma_{P_3}(x)/2\rangle : x\in E\}\\ &= \{\langle x, \max\{\rho_{P_1}(x)+\rho_{P_3}(x)/2, \rho_{P_2}(x)+\rho_{P_3}(x)/2\},\\ &\quad \min\{\sigma_{P_1}(x)+\sigma_{P_3}(x)/2, \sigma_{P_2}(x)+\sigma_{P_3}(x)/2\}\rangle : x\in E\}\\ &= \{\langle x, \rho_{P_2}(x)+\rho_{P_3}(x)/2, \sigma_{P_1}(x)+\sigma_{P_3}(x)/2\rangle : x\in E\} \qquad (6)\end{aligned}$$

From equation (5) and (6) proposes
$(P_1\cup P_2)@P_3 = (P_1@P_3)\cup(P_2@P_3)$.

Theorem 4. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1}), P_2 = (\rho_{P_2}, \sigma_{P_2})$ and $P_3 = (\rho_{P_3}, \sigma_{P_3})$ be three Pythagorean fuzzy sets. Then $P_1@(P_2 \cup P_3) = (P_1@P_2) \cup (P_1@P_3)$*

Proof: For three PFS A, B and C, From definition

$$\begin{aligned}
P_1@(P_2 \cup P_3) &= @\{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x \in E\} \\
&= \{\langle X, \max\{\rho_{P_2}(x), \rho_{P_3}(x)\}, \min\{\sigma_{P_2}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\
& Let \rho_{P_2} < \rho_{P_3}, \text{ and } \sigma_{P_2}(x) < \sigma_{P_3}(x) \\
& @\{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x \in E\} \\
&= @\{\langle x, , \rho_{P_3}(x), \sigma_{P_2}(x)\rangle : x \in E\} \\
&= \{\langle x, \rho_{P_1}(x) + \rho_{P_3}(x)/2, \\
& \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\} \qquad (7)
\end{aligned}$$

Also,

$$\begin{aligned}
(P_1@P_2) \cup (P_1@P_3) &= \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x)/2, \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\} \\
& \cup\{\langle x, \rho_{P_1}(x) + \rho_{P_3}(x)/2.\sigma_{P_1}(x) + \sigma_{P_3}(x)/2\rangle : x \in E\} \\
&= \{\langle x, \max\{\rho_{P_1}(x) + \rho_{P_2}(x)/2, \rho_{P_1}(x) + \rho_{P_3}(x)/2\}, \\
& \min\{\sigma_{P_1}(x) + \sigma_{P_2}(x)/2, \sigma_{P_1}(x) + \sigma_{P_3}(x)/2\}\rangle : x \in E\} \\
&= \{\langle x, \rho_{P_1}(x) + \rho_{P_3}(x)/2, \\
& \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\} \qquad (8)
\end{aligned}$$

From equation (7) and (8) yields
$P_1@(P_2 \cup P_3) = (P_1@P_2) \cup (P_1@P_3)$.

Theorem 5. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1}), P_2 = (\rho_{P_2}, \sigma_{P_2})$ and $P_3 = (\rho_{P_3}, \sigma_{P_3})$ be three Pythagorean fuzzy sets. Then $(P_1@P_2).P_3 = P_1.P_3@P_2.P_3$*

Proof: For three PFS A, B and C, From definition

$$\begin{aligned}
(P_1@P_2).P_3 &= \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x)/2, \\
& \sigma_{P_1}(x) + \sigma_{P_2}(x)/2\rangle : x \in E\}, \{\langle x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\
&= \{\langle x, \rho_{P_3}(x).[\rho_{P_1}(x) + \rho_{P_2}(x)]/2, \\
& [\sigma_{P_1}(x) + \sigma_{P_2}(x)]/2 + \sigma_C - [\sigma_{P_1}(x) + \sigma_{P_2}(x)]/2.\sigma_{P_3}(x)\rangle : x \in E\} \qquad (9)
\end{aligned}$$

Also,

$$\begin{aligned}
P_1.P_3@P_2.P_3 &= \{\langle x, \rho_{P_1}(x).\rho_{P_3}(x), \sigma_{P_1}(x) + \sigma_{P_3}(x) - \sigma_{P_1}(x).\sigma_{P_3}(x)\rangle : x \in E\} \\
& @\{\langle x, \rho_{P_2}(x).\rho_{P_3}(x), \sigma_{P_2}(x) + \sigma_{P_3}(x) - \sigma_{P_2}(x).\sigma_{P_3}(x)\rangle : x \in E\} \\
&= \{\langle x, [\rho_{P_1}(x).\rho_{P_3}(x) + \rho_{P_2}(x).\rho_{P_3}(x)]/2, \\
& \sigma_{P_1}(x) + \sigma_{P_3}(x) - \sigma_{P_1}(x).\sigma_{P_3}(x)\sigma_{P_2}(x) + \sigma_{P_3}(x) - \sigma_{P_2}(x).\sigma_{P_3}(x)/2\rangle : x \in E\} \\
&= \{\langle x, [\rho_{P_1}(x) + \rho_{P_2}(x)]/2.\rho_{P_3}(x), \\
& \{\sigma_{P_1}(x) + \sigma_{P_2}(x) + \sigma_{P_3}(x) - \sigma_{P_3}(x)[\sigma_{P_1}(x) + \sigma_{P_2}(x)]\}/2\rangle : x \in E\} \\
&= \{\langle x, [\rho_{P_1}(x) + \rho_{P_2}(x)]/2\}.\rho_{P_3}(x), \\
& [\sigma_{P_1}(x) + \sigma_{P_2}(x)]/2 + \sigma_{P_3}(x) - \sigma_{P_3}(x)[\sigma_{P_1}(x) + \sigma_{P_2}(x)]/2\rangle : x \in E\} \qquad (10)
\end{aligned}$$

From equation (9) and (10) presents
$(P_1@P_2).P_3 = P_1.P_3@P_2.P_3$.

Theorem 6. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1}), P_2 = (\rho_{P_2}, \sigma_{P_2})$ and $P_3 = (\rho_{P_3}, \sigma_{P_3})$ be three Pythagorean fuzzy sets, then $(P_1 \cap P_2).P_3 = (P_1.P_3) \cap (P_2.P_3)$*

Proof: Let $\rho_{P_1}(x) < \rho_{P_2}(x)$ and $\sigma_{P_1}(x) > \sigma_{P_2}(x)$, then

$$\begin{aligned}(P_1 \cap P_2).P_3 &= \{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x \in E\}.\{\langle x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x).\rho_{P_3}(x).\sigma_{P_1}(x) + \sigma_{P_3}(x) - \sigma_{P_1}(x)\sigma_{P_3}(x)\rangle : x \in E\} \end{aligned} \tag{11}$$

Also

$$\begin{aligned}(P_1.P_3) \cap (P_2.P_3) &= \{\langle x, \rho_{P_1}(x), \sigma_{P_1}(x)\rangle : x \in E\}.\{\langle x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\ &\quad \cap\{\langle x, \rho_{P_1}(x), \sigma_{P_2}(x)\rangle : x \in E\}.\{\langle : x, \rho_{P_3}(x), \sigma_{P_3}(x)\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x)\rho_{P_3}(x), \sigma_{P_1}(x) + \sigma_{P_3}(x) - \sigma_{P_1}(x)\sigma_{P_3}(x)\rangle : x \in E\} \\ &\quad \cap\{\langle x, \rho_{P_2}(x)\rho_{P_3}(x), \sigma_{P_2}(x) + \sigma_{P_3}(x) - \sigma_{P_2}(x)\sigma_{P_3}(x)\rangle : x \in E\} \\ &= \{\langle x, \min\{\rho_{P_1}(x)\rho_{P_3}(x).\rho_{P_2}(x)\rho_{P_3}(x)\} \\ &\quad \max\{\sigma_{P_1}(x)\sigma_{P_3}(x) - \sigma_{P_1}(x)\sigma_{P_3}(x), \sigma_{P_2}(x) + \sigma_{P_3}(x) - \sigma_{P_2}(x)\sigma_{P_3}(x)\}\rangle : x \in E\} \\ &\quad \text{Since} \rho_{P_1}(x) < \rho_{P_2}(x), \sigma_{P_1}(x) > \sigma_{P_2}(x) \\ &= \{\langle x, \rho_{P_1}(x)\rho_{P_3}(x), \sigma_{P_1}(x) + \sigma_{P_3}(x) - \sigma_{P_1}(x)\sigma_{P_3}(x) : x \in E\} \end{aligned} \tag{12}$$

From equation (11) and (12) gives
Therefore, $(P_1 \cap P_2).P_3 = (P_1.P_3) \cap (P_2.P_3)$.

Theorem 7. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1})$ and $P_2 = (\rho_{P_2}, \sigma_{P_2})$ be two Pythagorean fuzzy sets. Then $(P_1 \cap P_2) + (P_1 \cup P_2) = P_1 + P_2$*

Proof: P_1 and P_2 be two PFSs, then

$$\begin{aligned}(P_1 \cap P_2) + (P_1 \cup P_2) &= \{\langle x, \min\{\rho_{P_1}(x).\rho_{P_2}(x)\}, \max\{\sigma_{P_1}(x).\sigma_{P_2}(x)\}\rangle : x \in E\} \\ &\quad +\{\langle x, \max\{\rho_{P_1}(x).\rho_{P_2}(x)\}, \min\{\sigma_{P_1}(x).\sigma_{P_2}(x)\}\rangle : x \in E\} \\ &\quad Let \rho_{P_1}(x) < \rho_{P_2}(x) \text{ and } \sigma_{P_1}(x) < \sigma_{P-2}(x) \\ &= \{\langle x, \rho_{P_1}(x).\sigma_{P_2}(x)\rangle : x \in E\} + \{\langle x, \rho_{P_2}(x).\sigma_{P_1}(x)\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x) + \rho_{P_2}(x) - \rho_{P_1}(x)\rho_{P_2}(x), \sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x \in E\} \\ &= P_1 + P_2 \text{ by definition of"} + \text{"} \end{aligned}$$

As a result, $(P_1 \cap P_2) + (P_1 \cup P_2) = P_1 + P_2$.

Theorem 8. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1})$ and $P_2 = (\rho_{P_2}, \sigma_{P_2})$ be two Pythagorean fuzzy sets. Then $(P_1 \cap P_2).(P_1 \cup P_2) = P_1.P_2$*

Proof: A and B be two PFSs, then

$$\begin{aligned}(P_1 \cap P_2).(P_1 \cup P_2) &= \{\langle x, \min\{\rho_{P_1}(x), \rho_{P_2}(x)\}, \max\{\sigma_{P_1}(x), \sigma_{P_2}(x)\}\rangle : x \in E\}. \\ &\quad \{\langle x, \max\{\rho_{P_1}(x), \rho_{P_2}(x)\}, \min\{\sigma_{P_1}(x), \sigma_{P_2}(x)\}\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x), \sigma_{P_2}(x)\rangle : x \in E\}.\{\langle x, \rho_{P_2}(x), \sigma_{P_1}(x)\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x)\rho_{P_2}(x), \sigma_{P_2}(x) + \sigma_{P_1}(x) - \sigma_{P_2}(x)\sigma_{P_1}(x)\rangle : x \in E\} \\ &= \{\langle x, \rho_{P_1}(x)\rho_{P_2}(x), \sigma_{P_1}(x) + \sigma_{P_2}(x) - \sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x \in E\} \\ &= P_1.P_2 \text{ by definition of"."} \end{aligned}$$

Consequently, $(P_1 \cap P_2).(P_1 \cup P_2) = P_1.P_2$.

Theorem 9. *Let $P_1 = (\rho_{P_1}, \sigma_{P_1})$ and $P_2 = (\rho_{P_2}, \sigma_{P_2})$ be two Pythagorean fuzzy sets. Then $(P_1 + P_2)@(P_1.P_2) = P_1@P_2$*

Proof: P_1 and P_2 be two PFSs, then

$$\begin{aligned}(P_1+P_2)@(P_1.P_2) &= \{\langle x,\rho_{P_1}(x)+\rho_{P_2}(x)-\rho_{P_1}(x).\rho_{P_2}(x),\sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x\in E\}\\ &\quad @\{\langle x,\rho_{P_1}(x).\rho_{P_2}(x).\sigma_{P_1}(x)+\sigma_{P_2}(x)-\sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x\in E\}\\ &= \{\langle ,\{\rho_{P_1}(x)+\rho_{P_2}(x)-\rho_{P_1}(x)\rho_{P_2}(x)+\rho_{P_1}(x)\rho_{P_2}(x)\}/2,\\ &\quad \{\sigma_{P_1}(x).\sigma_{P_2}(x)+\sigma_{P_1}(x)+\sigma_{P_2}(x)-\sigma_{P_1}(x)\sigma_{P_2}(x)\}/2\rangle : x\in E\}\\ &= \{\langle x,\rho_{P_1}(x)+\rho_{P_2}(x)/2,\sigma_{P_1}(x)+\sigma_{P_2}(x)/2\rangle : x\in E\}\\ &= P_1@P_2 \text{ by definition.}\end{aligned}$$

Thus, $(P_1+P_2)@(P_1.P_2)=P_1@P_2$.

Theorem 10. *Let $P_1=(\rho_{P_1},\sigma_{P_1})$ and $P_2=(\rho_{P_2},\sigma_{P_2})$ be two Pythagorean fuzzy sets. Then $(P_1\cap P_2)@(P_1\cup P_2)=P_1@P_2$*

Proof: P_1 and P_2 be two PFSs, then

$$\begin{aligned}(P_1\cap P_2)@(P_1\cup P_2) &= \{\langle x,\min\{\rho_{P_1}(x),\rho_{P_2}(x)\},\max\{\sigma_{P_1}(x),\sigma_{P_2}(x)\}\rangle : x\in E\}\\ &\quad @\{\langle x,\max\{\rho_{P_1}(x),\rho_{P_2}(x)\},\min\{\sigma_{P_1}(x),\sigma_{P_2}(x)\}\rangle : x\in E\}\\ &\quad Let\rho_{P_1}(x)<\rho_{P_2}(x) \text{ and } \sigma_{P_1}(x)<\sigma_{P_2}(x)\\ &= \{\langle X,\rho_{P_1}(x),\sigma_{P_2}(x)\rangle/x\in E\}@\{\langle x,\rho_{P_2}(x).\sigma_{P_1}(x)\rangle : x\in E\}\\ &= \{\langle x,[\rho_{P_1}(x)+\rho_{P_2}(X)]/2,[\sigma_{P_2}(X)+\sigma_{P_1}(x)]/2\rangle : x\in E\}\\ &= P_1@P_2 \text{ by definition}\end{aligned}$$

Therefore, $(P_1\cap P_2)@(P_1\cup P_2)=P_1@P_2$.

Theorem 11. *Let $P_1=(\rho_{P_1},\sigma_{P_1})$ and $P_2=(\rho_{P_2},\sigma_{P_2})$ be two Pythagorean fuzzy sets. Then $(P_1\cap P_2)@(P_1\cup P_2)=(P_1+P_2)@(P_1.P_2)$*

Proof: Let P_1 and P_2 be two PFSs, then

$$\begin{aligned}(P_1\cap P_2)@(P_1\cup P_2) &= \{\langle x,\min\{\rho_{P_1}(x),\rho_{P_2}(x)\},\max\{\sigma_{P_1}(x),\sigma_{P_2}(x)\}\rangle : x\in E\}\\ &\quad @\{\langle x,\max\{\rho_{P_1}(x),\rho_{P_2}(x)\},\min\{\sigma_{P_1}(x),\sigma_{P_2}(x)\}\rangle : x\in E\}\\ &\quad Let\rho_{P_1}(x)<\rho_{P_2}(x) \text{ and } \sigma_{P_1}(x)<\sigma_{P_2}(x)\\ &= \{\langle X,\rho_{P_1}(x),\sigma_{P_2}(x)\rangle/x\in E\}@\{\langle x,\rho_{P_2}(x).\sigma_{P_1}(x)\rangle : x\in E\}\\ &= \{\langle x,[\rho_{P_1}(x)+\rho_{P_2}(X)]/2,[\sigma_{P_2}(X)+\sigma_{P_1}(x)]/2\rangle : x\in E\}\\ &= P_1@P_2 \text{ by definition}\end{aligned} \tag{13}$$

$$\begin{aligned}(P_1+P_2)\ @\ (P_1.P_2) &= \{\langle x,\rho_{P_1}(x)+\rho_{P_2}(x)-\rho_{P_1}(x).\rho_{P_2}(x),\sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x\in E\}\\ &\quad @\{\langle x,\rho_{P_1}(x).\rho_{P_2}(x).\sigma_{P_1}(x)+\sigma_{P_2}(x)-\sigma_{P_1}(x)\sigma_{P_2}(x)\rangle : x\in E\}\\ &= \{\langle ,\{\rho_{P_1}(x)+\rho_{P_2}(x)-\rho_{P_1}(x)\rho_{P_2}(x)+\rho_{P_1}(x)\rho_{P_2}(x)\}/2,\\ &\quad \{\sigma_{P_1}(x).\sigma_{P_2}(x)+\sigma_{P_1}(x)+\sigma_{P_2}(x)-\sigma_{P_1}(x)\sigma_{P_2}(x)\}/2\rangle : x\in E\}\\ &= \{\langle x,\rho_{P_1}(x)+\rho_{P_2}(x)/2,\sigma_{P_1}(x)+\sigma_{P_2}(x)/2\rangle : x\in E\}\\ &= P_1@P_2 \text{ by definition.}\end{aligned} \tag{14}$$

Hence, $(P_1\cap P_2)@(P_1\cup P_2)=(P_1+P_2)@(P_1.P_2)$.

4|Conclusion

This paper provide a comprehensive understanding of various operations on Pythagorean fuzzy sets, offering insights into their theoretical foundations, computational aspects, and practical implications. By elucidating these operations, we seek to contribute to the advancement of decision support systems and intelligent methodologies capable of handling uncertainty in a more nuanced and effective manner.

In future, we will discuss the potential applications of these operations in fields such as engineering, finance, medicine, and artificial intelligence, highlighting the practical utility and versatility of Pythagorean fuzzy sets in addressing complex decision-making problems.

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

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References

[1] Abdullah, S., Naeem, M., Imran, M. (July 2013). On intuitionistic fuzzy σ-ideals of σ-LA-semigroups. *Annals of Fuzzy Mathematics and Informatics*. 6(1), 17-31.

[2] A. Abou-Zaid. (1989). On fuzzy subnear-ring. *Fuzzy Sets and Systems*. 81, 383-393.

[3] Adak, A. K., Bhowmik, M. (2011). Interval cut-set of interval-valued intuitionistic fuzzy sets. *African Journal of Mathematics and Computer Sciences*. 4(4), 192-200.

[4] Adak, A. K., Bhowmik, M. and Pal, M. Interval cut-set of generalized interval-valued intuitionistic fuzzy sets. *International Journal of Fuzzy System Applications*. 2 (3) (2012) 35-50.

[5] Adak, A. K., Salokolaei, D. D. (2021). Some Properties Rough Pythagorean Fuzzy Sets. *Fuzzy Information and Engineering (IEEE)* , 13(4) 420-435.

[6] Adak, A. K.,& Kumar, D. (2022). Some properties of Pythagorean fuzzy ideals of Γ- near rings. *Palestine Journal of Mathematics*. 11(4), 336-346.

[7] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*. 20, 87-96.

[8] Biswas, R. (1990). Fuzzy subgroups and anti-fuzzy subroups. *Fuzzy Sets and Sys*. 35(1), 121-124.

[9] Davvaz, B. (2008). Fuzzy R-subgroups with thresholds of near-rings and implication operators. *Soft Computing*. 12, 875-879.

[10] Ebrahimnejad, A., Adak, A. K., Jamkhaneh, E.B. (2019). Eigenvalue of Intuitionistic Fuzzy Matrices Over Distributive Lattice. *International Journal of Fuzzy System Applications (IJFSA)*. 8(1), 1-18.

[11] Garg, H. (2016). A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. *J Intell Fuzzy Syst*. 31(1), 529-540.

[12] Garg, H. (2016). A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. *Int J Intell Syst*. 31(9), 886-920.

[13] Garg, H. (2017). Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process. *Int J Intell Syst*. 32(6), 597-630.

[14] Jun, Y. B., Kim, K. H., Yon, Y. H. (1999). Intuitionistic fuzzy ideals of near-rings. *J. Inst. Math. Comp. Sci.* 12 (3), 221-228.

[15] Jun, Y. B. (2002). Interval-valued fuzzy R-subgroups of near-rings. *Indian J. pure appl. Math*. 33(1), 71-80.

[16] Kim, K. H. Jun, Y. B. (2002). On fuzzy R-subgroups of near-rings. *J. Fuzzy Math.*. 8, 549-558.

[17] Kuncham, S.P., Bhavanari, S. (2005). Fuzzy prime ideal of a gamma near ring. *Soochow Journal of Mathematics*. 31 (1), 121-129.

[18] Ma, X., Zhan, J. (2009). On generalized fuzzy R-subgroups of near-rings. *International Workshop on Intelligent Systems and Applications*. 1724-1727.

[19] Prince Williams, D. R., Latha, K. B., Chandrasekar, E. (March 2013). T_G-interval-valued $(\in, \in, \vee q)$-fuzzy subgroups, *Annals of Fuzzy Mathematics and Informatics*. 5(2), 283-300.

[20] Roh, E. H. Roh, Kim, K. H., Lee, J. G., (2006). Intuitionistic Q-Fuzzy Subalgebras of BCK/BCI-Algebras. *International Mathematical Forum*. 1 (24), 1167-1174.

[21] Yager, R. R., Abbasov, A. M. (2013). Pythagorean membeship grades, complex numbers and decision making. *Int J Intell Syst*. 28, 436-452.

[22] R. R. Yager, R. R. (2013). Pythagorean fuzzy subsets. *In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting*. Edmonton, Canada. 57-61.

[23] R. R. Yager, R. R. (2014). Pythagorean membership grades in multicriteria decision making. *IEEE Transaction on Fuzzy Systems*. 22, 958-965.

[24] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338-353.

[25] Zhang, X., Xu, Z.(2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. *Int J Intell Syst* . 29(12), 1061-1078.